1. Determine which amounts of postage can be formed using just 3-cent and 10-cent stamps. Prove your answer by using strong induction.

We will prove this statement using strong induction.
Base case: We can form postage of 18, 19, and 20 cents using six 3-cent stamps, three 3-cent and one 10-cent stamp, and two 10-cent stamps.
Inductive step: Our inductive hypothesis is that $P(j)$ is true for $18 \leq j \leq k$, where $k \in \mathbb{Z}$, $k \geq 20$. We want to show that $P(k + 1)$ is true, which is to say we can form postage of $k + 1$ cents.
Using the inductive hypothesis, we assume $P(k - 2)$ is true because $k - 2 \geq 18$, and so we can form postage of $k - 2$ cents using only 3-cent and 10-cent stamps. To form postage of $k + 1$ cents, we need only add another 3-cent stamp to the stamps we used to form $k - 2$ cents. Therefore, by strong mathematical induction, $P(k + 1)$ is true whenever $P(k - 2)$ is true. Q.E.D.

2. Use strong induction to show that every positive integer $n$ can be written as the sum of distinct powers of two, that is, as a sum of a subset of integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. Hint: For the inductive step, separately consider the case where $k + 1$ is even and when it is odd. When it is even, note that $\frac{k+1}{2}$ is an integer.

We will prove this statement using strong induction on $n$. Let $n$ be a positive integer.
We will prove that $n$ can be written as the sum of distinct powers of 2. Let $P(n)$ be the statement that we can write $n$ as the sum of distinct powers of 2.
Base base: $P(1)$ is true because $1 = 2^0$.
Inductive step: Assume for $1 \leq j \leq k$ for some integer $k$ that $P(k)$ is true. That is, assume we can write all integers from 1 to $k$ as the sum of distinct primes. We have two cases to consider.

Case 1: $k$ is even. Then, $k + 1$ is odd. Now, by the inductive hypothesis, we have that $k$ has an expansion

$$k = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_n}$$

where $m_1 > m_2 > \ldots > m_n$. (Note: we need to write it in general terms because we don’t know which powers of 2 are needed for the integer. There is no reason that $k$ would have to have 2 or 4 or any other particular power of 2 in it’s expansion.). Notice that since $k$ is even, there is no $2^0$ in the expansion; so, from the expansion for $k$ we add $2^0$ to yield

$$k = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_n} + 2^0$$
to give the expansion for \( k_1 \).

Case 2: \( k \) is odd. Then \( k + 1 \) is even. (Note: we can’t do the same as the other case because we already have a \( 2^0 \) in the expansion of \( k \) and so a second one would mean we would have to change it to \( 2^1 \) to make it unique, but then if we already have a \( 2^1 \) in the expansion, we have to change that to a \( 2^2 \), which we may already have, and since we don’t actually see the expansion for a generic \( k \), we can’t begin to approach this way). Since \( k + 1 \) is even, then \( \frac{k+1}{2} \in \mathbb{Z} \) and therefore by the inductive hypothesis, has an expansion consisting of the sum of distinct powers of 2. Let the expansion be

\[
\frac{k + 1}{2} = 2^{m_1} + 2^{m_2} + \ldots + 2^{m_n}
\]

Then, we can multiply both sides by 2, yielding

\[
2 \left( \frac{k + 1}{2} \right) = 2(2^{m_1} + 2^{m_2} + \ldots + 2^{m_n})
\]

\[
k + 1 = 2^{m_1+1} + 2^{m_2+1} + \ldots + 2^{m_n+1}
\]

which completes the inductive step. So, by strong induction, we have shown that if \( P(j) \) is true for \( 1 \leq j \leq k \), then \( P(k + 1) \) is true as well. Q.E.D.