The Symmetry Transformations of the Square

There are 8 different symmetry transformations for the square. They are drawn below. The differences between the transformations are shown by the new positions of the letters that represent the vertices, and are also described in words.

Rotations

E : This is the ‘stay-put’ transformation. E is called the identity.

U : This rotates the square through a quarter turn.

V : This rotates the square through a half turn.

W : This rotates the square through a three-quarters turn.

Each rotation is about the center of the square and is drawn counter-clockwise. This is because mathematicians consider this direction to be positive.
Reflections

P : This reflects about a vertical mirror line through the center.

Q : This reflects about a horizontal mirror through the center.

R : This reflects about the diagonal mirror line through AC.

S : This reflects about the diagonal mirror line through BD

You can operate on two symmetries by performing them under composition. Complete the following table. To fill in a square, first perform the operation in the vertical column and then the operation in the horizontal row. For example, to fill in the fourth open square in the last row, we would first perform the operation S on the identity square and then the operation W on the resulting square. This would be written as \((W \circ S)(E)\) or \(W(S(E))\).

<table>
<thead>
<tr>
<th></th>
<th>E</th>
<th>U</th>
<th>V</th>
<th>W</th>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>E</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>U</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>V</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>W</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Q</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>R</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>S</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Groups

A mathematical system consisting of a set \(G\) and an operation that follows a few simple rules is called a group. These rules are:

1. **Closure**: If \(a\) and \(b\) are elements of the group \(G\) then so is \(a \ast b\).

2. **Associativity**: If \(a\), \(b\) and \(c\) are elements of the group \(G\) then \((a \ast b) \ast c = a \ast (b \ast c)\).

3. **Identity**: There is an element \(e\) of the group \(G\) such that for any element \(a\) of the group, \(a \ast e = a = e \ast a\).

4. **Inverse**: For any element \(a\) of the group \(G\) there is an element \(a^{-1}\) such that \(a \ast a^{-1} = e\) and \(a^{-1} \ast a = e\).

The study of systems that obey these four rules is the basis of **Group Theory**.

Based on the table you filled out on the other sheet, answer the following questions.

1. Is the set of symmetries of the square under the operation of composition a group?
2. Is the set of rotation symmetries of the square under composition a group?
3. Is the set of reflection symmetries of the square under composition a group?
4. What are the inverses of each element?