§ 3 Isomorphic Binary Structures

2-8,16-18,26,27

2. \(<\mathbb{Z}, +>\) with \(<\mathbb{Z}, +>\) where \(\phi(n) = -n\) for \(n \in \mathbb{Z}\). \textbf{YES}
   \[\phi(a + b) = -a - b\] and \(\phi(a) + \phi(b) = -a - b\)

3. \(<\mathbb{Z}, +>\) with \(<\mathbb{Z}, +>\) where \(\phi(n) = 2n\) for \(n \in \mathbb{Z}\). \textbf{NO}
   \(\phi\) is not onto \(\mathbb{Z}\). There is no \(n \in \mathbb{Z} \ni \phi(n) = 1 \in \mathbb{Z}'\).

4. \(<\mathbb{Z}, +>\) with \(<\mathbb{Z}, +>\) where \(\phi(n) = n + 1\) for \(n \in \mathbb{Z}\). \textbf{NO}
   \(\phi(a + b) = a + b + 1\) but \(\phi(a) + \phi(b) = a + b + 2\)

5. \(<\mathbb{Q}, +>\) with \(<\mathbb{Q}, +>\) where \(\phi(x) = \frac{x}{2}\) for \(x \in \mathbb{Q}\). \textbf{YES}
   \(\phi(a + b) = \frac{a + b}{2}\) and \(\phi(a) + \phi(b) = \frac{a}{2} + \frac{b}{2}\)

6. \(<\mathbb{Q}, \cdot>\) with \(<\mathbb{Q}, \cdot>\) where \(\phi(x) = x^2\) for \(x \in \mathbb{Q}\). \textbf{NO}
   Not 1-1: \(\phi(a) = \phi(-a)\) but \(a \neq -a\)

7. \(<\mathbb{R}, \cdot>\) with \(<\mathbb{R}, \cdot>\) where \(\phi(x) = x^3\) for \(x \in \mathbb{R}\). \textbf{YES}
   \(\phi(ab) = (ab)^3\) and \(\phi(a) \cdot \phi(b) = a^3 \cdot b^3 = (ab)^3\)

8. \(<M_2(\mathbb{R}), \cdot>\) with \(<\mathbb{R}, \cdot>\) where \(\phi(A)\) is the determinant of the matrix \(A\). \textbf{NO}
   Not 1-1.
   \[A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \quad |A| = |B| \text{ but } A \neq B.\]

16. The map \(\phi : \mathbb{Z} \to \mathbb{Z}\) defined by \(\phi(n) = n + 1\) for \(n \in \mathbb{Z}\) is one to one and onto \(\mathbb{Z}\). Give the definition of a binary operation \(\ast\) on \(\mathbb{Z}\) such that \(\phi\) is an isomorphism mapping.

a. \(<\mathbb{Z}, +>\) onto \(<\mathbb{Z}, \ast>\)
   For \(\phi\) to be an isomorphism, we must have
   \[m \ast n = \phi(m - 1) \ast \phi(n - 1) = \phi((m - 1) + (n - 1)) = \phi(m + n - 2) = m + n - 1.\]
   The identity element \(\phi(0) = 1\).

b. \(<\mathbb{Z}, \ast>\) onto \(<\mathbb{Z}, +>\)
   Using the fact that \(\phi^{-1}\) is an isomorphism, we must have
   \[m \ast n = \phi^{-1}(m + 1) \ast \phi^{-1}(n + 1) = \phi^{-1}((m + 1) + (n + 1)) = \phi^{-1}(m + n + 2) = m + n + 1.\]
   The identity element is \(\phi^{-1}(0) = -1\).

17. The map \(\phi : \mathbb{Z} \to \mathbb{Z}\) defined by \(\phi(n) = n + 1\) for \(n \in \mathbb{Z}\) is one to one and onto \(\mathbb{Z}\). Give the definition of a binary operation \(\ast\) on \(\mathbb{Z}\) such that \(\phi\) is an isomorphism mapping.

a. \(<\mathbb{Z}, \cdot>\) onto \(<\mathbb{Z}, \ast>\)
   For \(\phi\) to be an isomorphism, we must have
   \[m \ast n = \phi(m - 1) \ast \phi(n - 1) = \phi((m - 1) \cdot (n - 1)) = \phi(mn - m - n + 1) = mn - m - n + 2.\]
   The identity element is \(\phi(1) = 2\).

b. \(<\mathbb{Z}, \ast>\) onto \(<\mathbb{Z}, \cdot>\)
   Using the fact that \(\phi^{-1}\) must also be an isomorphism, we must have
   \[m \ast n = \phi^{-1}(m + 1) \ast \phi^{-1}(n + 1) = \phi^{-1}((m + 1) + (n + 1)) = \phi^{-1}(mn + m + n + 1) = mn + m + n.\]
   The identity element is \(\phi^{-1}(1) = 0\).
18. The map $\phi : Q \rightarrow Q$ defined by $\phi(x) = 3x - 1$ for $x \in Q$ is one to one and onto $Q$. Give the
definition of a binary operation $*$ on $Q$ such that $\phi$ is an isomorphism mapping.

a. $< Q, + >$ onto $< Q, * >$
   For $\phi$ to be an isomorphism, we must have
   $a * b = \phi \left( \frac{a+1}{3} \right) * \phi \left( \frac{b+1}{3} \right) = \phi \left( \frac{a+b+2}{3} \right) = a + b + 1$.
   The identity element is $\phi(0) = -1$.

b. $< Q, * >$ onto $< Q, + >$
   Using the fact that $\phi^{-1}$ must also be an isomorphism, we must have
   $a * b = \phi^{-1}(3a - 1) * \phi^{-1}(3b - 1) = \phi^{-1}((3a - 1) + (3b - 1)) = \phi^{-1}(3a + 3b - 2) = a + b - \frac{1}{3}$.
   The identity element is $\phi^{-1}(0) = \frac{1}{3}$.

26. Since $f$ is a bijection, $f^{-1}$ is a bijection also. It needs only to be shown that $f^{-1}$ is a
homomorphism.
Since $f$ is a homomorphism, we know for $a,b \in S$, $f(a) = x$ and $f(b) = y$ for $x,y \in S'$. We
also know that $f(a * b) = f(a) * f(b)$.
We want to show that $f^{-1}(x * y) = f^{-1}(x) * f^{-1}(y)$.
Consider the following:

\[
\begin{align*}
f^{-1}(x * y) &= f^{-1}(f(a) * f(b)) \\
&= f^{-1}(f(a * b)) \\
&= a * b \\
&= f^{-1}(f(a)) * f^{-1}(f(b)) \\
&= f^{-1}(x) * f^{-1}(y)
\end{align*}
\]

A second way to go:
1-1: Suppose $\phi^{-1}(a') = \phi^{-1}(b')$ for all $a', b' \in S'$. Then $a' = \phi(\phi^{-1}(a')) = \phi(\phi^{-1}(b')) = b'$.
So, $\phi^{-1}$ is 1-1.
Onto: Let $a \in S$. Then $\phi^{-1}(\phi(a)) = a$, so $\phi^{-1}$ maps $S'$ onto $S$.
Homomorphism Property: Let $a', b' \in S'$. Now,

\[
\phi(\phi^{-1}(a' * b')) = a' * b'
\]

Because $\phi$ is an isomorphism,

\[
\phi(\phi^{-1}(a') * \phi^{-1}(b')) = \phi(\phi^{-1}(a')) * \phi(\phi^{-1}(b')) = a' * b'
\]

also. Because $\phi$ is 1-1, we can conclude that the operation is preserved.

27. Onto: We know there is a $y \in S' \ni x \in S \leftrightarrow \phi(x) = y$. We also know there is a $z \in S'' \ni y \in S' \leftrightarrow \psi(y) = z$. So, $\exists z \in S''$ and $x \in S \ni \psi(\phi(x)) = z$.
1-1: For $x, y \in S$, $\phi(x) = \phi(y)$ only when $x = y$ and for $\phi(x), \phi(y) \in S'$, $\psi(\phi(x)) = \psi(\phi(y))$
only when $\phi(x) = \phi(y)$ which is only when $x = y$.
Homomorphism: We want to show $\psi(\phi(x * y)) = \psi(\phi(x)) *'' \psi(\phi(y))$.

$\psi(\phi(x * y)) = \psi(\phi(x) *' \phi(y)) = \psi(\phi(x)) *'' \psi(\phi(y))$. 

\]
Another way to go:

**1-1:** Let \( a, b \in S \) and suppose \((\psi \circ \phi)(a) = (\psi \circ \phi)(b)\). Then \((\psi(\phi(a)) = (\psi(\phi(b)))\). Because \(\psi\) is 1-1, we conclude that \(\phi(a) = \phi(b)\). Because \(\phi\) is 1-1, it must be so that \(a = b\).

**Onto:** Let \( a'' \in S''. \) Because \(\psi\) maps \(S'\) onto \(S''\), there exists \( a' \in S' \) such that \(\psi(a') = a''\). Because \(\psi\) maps \(S\) onto \(S'\), there exists \( a \in S \) such that \(\phi(a) = a'\). Then, \((\psi \circ \phi)(a) = (\psi(\phi(a)) = \psi(a') = a'',\) so \(\psi \circ \phi\) maps \(S\) onto \(S''\).

**Homomorphism:** Let \( a, b \in S \). Since \(\psi\) and \(\phi\) are isomorphisms, \((\psi \circ \phi)(a \ast b) = (\psi(\phi(a \ast b))) = (\psi(\phi(a)) \ast' \psi(\phi(b))) = \psi(\phi(a) \ast' \phi(b)) = (\psi(\phi(a)) \ast'' \psi(\phi(b)))\).