§ 10 Cosets and the Theorem of Lagrange

Homework: 1-7, 12-16, 26, 28-33

1. Find all cosets of the subgroup $4\mathbb{Z}$ of $\mathbb{Z}$.

$4\mathbb{Z} = \{\cdots, -8, -4, 0, 4, 8, \cdots\}$

$1 + 4\mathbb{Z} = \{\cdots, -7, -3, 1, 5, 9, \cdots\}$

$2 + 4\mathbb{Z} = \{\cdots, -6, -2, 2, 6, 10, \cdots\}$

$3 + 4\mathbb{Z} = \{\cdots, -5, -1, 3, 7, 11, \cdots\}$

2. Find all cosets of the subgroup $4\mathbb{Z}$ of $2\mathbb{Z}$.

$4\mathbb{Z} = \{\cdots, -8, -4, 0, 4, 8, \cdots\}$

$2 + 4\mathbb{Z} = \{\cdots, -6, -2, 2, 6, 10, \cdots\}$

3. Find all cosets of the subgroup $<2>$ of $\mathbb{Z}_{12}$.

$<2> = \{0, 2, 4, 6, 8, 10\}$

$1+<2> = \{1, 3, 5, 7, 9, 11\}$

4. Find all cosets of the subgroup $<4>$ of $\mathbb{Z}_{12}$.

$<4> = \{0, 4, 8\}$

$1+<4> = \{1, 5, 9\}$

$2+<4> = \{2, 6, 10\}$

$3+<4> = \{3, 7, 11\}$

5. Find all cosets of the subgroup $<18>$ of the group $\mathbb{Z}_{36}$.

$<18> = \{0, 18\}$, $1+<18> = \{1, 19\}$, $2+<18> = \{2, 20\}$, $3+<18> = \{3, 21\}$

6. Find all of the left cosets of the subgroup $\{\rho_0, \mu_2\}$ of the group $D_4$.

$\{\rho_0, \mu_2\}, \{\rho_1, \delta_1\}, \{\rho_2, \mu_1\}, \{\rho_3, \delta_1\}$

7. Repeat the preceding exercise, but find the right cosets this time. Are they the same as the left cosets?

$\{\rho_0, \mu_2\}, \{\rho_1, \delta_1\}, \{\rho_2, \mu_1\}, \{\rho_3, \delta_1\}$

No, they are not the same.

12. Find the index of $<3>$ in the group $\mathbb{Z}_{24}$.

$<3> = \{1, 3, 6, 9, 12, 15, 18, 21\}$ which has 8 elements, so the index $(\mathbb{Z}_{24} : <3>)$ is $\frac{24}{8} = 3$.

13. Find the index of $<\mu_1>$ in the group $S_3$.

$<\mu_1> = \{\rho_0, \mu_1\}$ has 2 elements, so it’s index $(S_3 : <\mu_1>)$ (the number of left cosets) is $\frac{6}{2} = 3$.

14. Find the index of $<\mu_2>$ in the group $D_4$.

$<\mu_2> = \{\rho_0, \mu_2\}$ has 2 elements, so it’s index $(D_4 : <\mu_2>)$ is $\frac{8}{2} = 4$. 
26. Prove that the relation ∼ is an equivalence relation.

15. Let σ = (1254)(23) in S_5. Find the index of <σ> in S_5.
   1: (1254)(23) = (12354)
   2: (12354)(12354) = (13425)
   3: (13425)(12354) = (15243)
   4: (15243)(12354) = (14532)
   5: (14532)(12354) = (1)
   So, the order of (1254)(23) is 5. There are 5! elements in S_5, so the index (S_5 : <σ>) is $\frac{5!}{5} = 4!$.

   (1245)(36)(1245)(36) = (15)(24), so the order of μ is 4. So, the index (S_6 : <μ>) is $\frac{6!}{4} = 180$.

26. Prove that the relation ∼_R of Theorem 10.1 is an equivalence relation.
   From the statement of the theorem, a ∼_R b ↔ ab^{-1} ∈ H. We will use this to show the three required properties for ∼_R to be an equivalence relation.

   Reflexive: Let a ∈ G. Then, aa^{-1} = e and e ∈ h because H is a subgroup. So, a ∼_R a.

   Symmetric: Suppose a ∼_R b. Then, ab^{-1} ∈ H. Because H is a subgroup, the inverse \((ab^{-1})^{-1} = ba^{-1} ∈ H\), so b ∼_R a.

   Transitive: Suppose a ∼_R b and b ∼_R c. Then ab^{-1} and bc^{-1} are both in H. Because H is a subgroup, it is closed, so \((ab^{-1})(bc^{-1}) = ac^{-1} ∈ H\), so a ∼_R c.

26. Let H be a subgroup of a group G. Prove that if the partition of G into left cosets of H is the same as the partition into right cosets of H, then g^{-1}hg ∈ H for all g ∈ G and all h ∈ H. Show that every left coset gH is the same as the right coset Hg.
   We will show that gH = Hg by showing that each coset is a subset of the other. Let gh ∈ gH where g ∈ G and h ∈ H. Then gh = ghg^{-1}g = [(g^{-1})^{-1}hg^{-1}g], which is in Hg because \((g^{-1})^{-1}hg^{-1}\) is in H by hypothesis. So, gH is a subset of Hg.
   Now let hg ∈ Hg where g ∈ G and h ∈ H. Then hg = gg^{-1}hg = g(g^{-1}hg) is in Hg because g^{-1}hg is in H by hypothesis. So, Hg is a subset of gH also, so gH − Hg.

29. Let H be a subgroup of a group G. Prove that if the partition of G into left cosets of H is the same as the partition into right cosets of H, then g^{-1}hg ∈ H for all g ∈ G and h ∈ H. (Note that this is the converse of the previous exercise).
   Let h ∈ H and g ∈ G. By hypothesis, Hg = ghg. So, hg = gh_1 for some h_1 ∈ H. Then, g^{-1}hg = h_1, showing that g^{-1}hg ∈ H.

30. If aH = bH then Ha = Hb.
    The statement is false. Let G = S − 3, H = {ρ_0, μ_1}, a = ρ_1 and b = μ_3. Then aH = {ρ_1, μ_3} = bH, but Ha = {ρ_1, μ_2} while Hb = {ρ_2, μ_3}.

31. If Ha = Hb then b ∈ Ha.
    The statement is true. b = eb and e ∈ H, so b ∈ Hb. Because b ∈ Hb and Hb = Ha, b ∈ Ha.

32. If aH = bH, then Ha^{-1} = Hb^{-1}.
    The statement is true. Because H is a subgroup, we have \{h^{-1} | h ∈ H\} = H. Therefore, Ha^{-1} = \{ha^{-1} | h ∈ H\} = \{h^{-1}a^{-1} | h ∈ H\} = \{(ah)^{-1} | h ∈ H\}. That is, Ha^{-1} consists of all inverses of elements in aH. Similarly, Hb^{-1} consists of all inverses of elements in bH. Because aH = bH, we must have that Ha^{-1} = Hb^{-1}. 

33. If $aH = bH$, then $a^2H = b^2H$.
The statement is false. Let $H$ be the subgroup $\{\rho_0, \mu_2\}$ of $D_4$. Then $\rho_1H = \delta_2H = \{\rho_1, \delta_2\}$ and $\rho_1^2H = \rho_2H = \{\rho_2, \mu_1\}$ but $\delta_2^2H = \rho_0H = H = \{\rho_0, \mu_2\}$.

Another example: Let $H = \{(1), (12)\}$, $a = (13)$ and $b = (123)$. Consider:

$$aH = (13)(1, (12)) = \{(13), (123)\}$$

$$bH = (123)(1, (12)) = \{(123), (13)\}$$

So, $aH = bH$. But, we also have

$$a^2H = (13)(13)(1, (12)) = H$$

and

$$b^2H = (123)(123)(1, (12)) = (132)(1, (12)) = \{(132), (23)\}$$

So, even though $aH = bH$, $a^2H \neq b^2H$. 