7.4 Relative Rates of Growth
This section is about comparing functions to see which dominate as \( x \to \infty \).

**Definition**

Let \( f(x) \) and \( g(x) \) be positive for some sufficiently large \( x \).  
1. \( f \) grows faster than \( g \) as \( x \to \infty \) if
   \[
   \lim_{x \to \infty} \frac{f(x)}{g(x)} = \infty
   \]
2. \( f \) grows slower than \( g \) as \( x \to \infty \) if
   \[
   \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0
   \]
3. \( f \) grows at the same rate as \( g \) as \( x \to \infty \) if
   \[
   \lim_{x \to \infty} \frac{f(x)}{g(x)} = L
   \] where \( L \) is finite and positive.
Domination

This section is about comparing functions to see which dominate as $x \to \infty$.

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3. $f$ grows at the same rate as $g$ as $x \to \infty$ if
   
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Example
Which grows faster: $e^x$ or $x$?
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Hint: Use L’Hospital’s Rule
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Which grows faster: $e^x$ or $x$?

Hint: Use L’Hopital’s Rule

$e^x$ grows faster than $x$
Which Grows Faster?

Example

Which grows faster: $e^x$ or $x^{20}$?
Which Grows Faster?

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Which grows faster: $e^x$ or $x^{20}$?

$e^x$ grows faster than $x^{20}$
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Which grows faster: $e^x$ or $x^{20}$?

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Can we generalize this?
Example

Which grows faster: $2^x$ or $4^x$?
Which Grows Faster?

Example

Which grows faster: $2^x$ or $4^x$?

$2^x$ grows slower than $4^x$
Which Grows Faster?

Example

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$2^x$ grows slower than $4^x$

Can we generalize this?
Which Grows Faster?

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Which grows faster: \( \ln(x) \) or \( x \)?
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Which grows faster: \( \log_3 x \) or \( \log_2 x \)?
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Which grows faster: \( \log_3 x \) or \( \log_2 x \)?

\( \log_3(x) \) grows at the same rate as \( \log_2(x) \)
Example

Which grows faster: $\log_3 x$ or $\log_2 x$?

$log_3(x)$ grows at the same rate as $\log_2(x)$

$$\log_3(x) = \frac{\log(x)}{\log(3)}$$
Which Grows Faster?

Example

Which grows faster: \( \log_3 x \) or \( \log_2 x \)?

\( \log_3(x) \) grows at the same rate as \( \log_2(x) \)

\[
\log_3(x) = \frac{\log(x)}{\log(3)} \\
\log_2(x) = \frac{\log(x)}{\log(2)}
\]
Which grows faster: \( \log_3 x \) or \( \log_2 x \)?

\( \log_3(x) \) grows at the same rate as \( \log_2(x) \)

\[
\begin{align*}
\log_3(x) &= \frac{\log(x)}{\log(3)} \\
\log_2(x) &= \frac{\log(x)}{\log(2)}
\end{align*}
\]

\[
\lim_{x \to \infty} \frac{\log(x)}{\log(3)} = \frac{\log(x)}{\log(2)}
\]
Which Grows Faster?

Example

Which grows faster: \( \log_3 x \) or \( \log_2 x \)?

\( \log_3 (x) \) grows at the same rate as \( \log_2 (x) \)

\[
\begin{align*}
\log_3 (x) &= \frac{\log(x)}{\log(3)} \\
\log_2 (x) &= \frac{\log(x)}{\log(2)} \\
\lim_{x \to \infty} \frac{\log(x)}{\log(3)} &= \frac{\log(2)}{\log(3)}
\end{align*}
\]
Comparisons

Sometimes it is easier to show functions grow at the same rate by comparing them to a common function.
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Sometimes it is easier to show functions grow at the same rate by comparing them to a common function. Consider $\sqrt{x^2 + 1}$ vs. $\sqrt[3]{2x^3 + 1}$. Differentiating would be a pain, so we wouldn’t want to use L’Hopital’s Rule, but we could use something simple, like $x$.

$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} =$$
Comparisons

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$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \to \infty} \sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}} = 1$$
Sometimes it is easier to show functions grow at the same rate by comparing them to a common function. Consider $\sqrt{x^2 + 1}$ vs. $\sqrt[3]{2x^3} + 1$. Differentiating would be a pain, so we wouldn’t want to use L’Hopital’s Rule, but we could use something simple, like $x$.

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$$\lim_{x \to \infty} \frac{\sqrt[3]{2x^3} + 1}{x} = \sqrt[3]{2} + \frac{1}{x^3} \to 0$$
Sometimes it is easier to show functions grow at the same rate by comparing them to a common function. Consider $\sqrt{x^2 + 1}$ vs. $\sqrt[3]{2x^3 + 1}$. Differentiating would be a pain, so we wouldn’t want to use L’Hopital’s Rule, but we could use something simple, like $x$.

$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \to \infty} \sqrt{\frac{x^2 + 1}{x^2}} = 1$$

$$\lim_{x \to \infty} \frac{\sqrt[3]{2x^3 + 1}}{x} = \lim_{x \to \infty} \sqrt[3]{\frac{2x^3 + 1}{x^3}} = \sqrt[3]{2}$$

Since they both grow at the same rate, we conclude that $\sqrt{x^2 + 1}$ and $\sqrt[3]{2x^3 + 1}$ grow at the same rate.
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$$\lim_{x \to \infty} \frac{\sqrt{x^2 + 1}}{x} = \lim_{x \to \infty} \sqrt{\frac{x^2}{x^2} + \frac{1}{x^2}} = 1$$

$$\lim_{x \to \infty} \frac{\sqrt[3]{2x^3 + 1}}{x} = \lim_{x \to \infty} \sqrt[3]{\frac{2x^3}{x^3} + \frac{1}{x^3}} = \sqrt[3]{2}$$

Since they both grow at the same rate as the same function, we conclude that $\sqrt{x^2 + 1}$ and $\sqrt[3]{2x^3 + 1}$ grow at the same rate.
A function $f$ is of **smaller order** than $g$ as $x \to \infty$ if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 0$$

We indicate this by writing

$$f = o(g)$$

which is read “$f$ is little oh of $g$”.
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Example

$$x = o(e^x)$$
Definition

Let $f(x)$ and $g(x)$ be positive for sufficiently large $x$. Then $f$ is at most the order of $g$ as $x \to \infty$ if there is a positive integer $M$ for which

$$\frac{f(x)}{g(x)} \leq M$$

for $x$ sufficiently large, We indicate this by writing

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### Definition

Let $f(x)$ and $g(x)$ be positive for sufficiently large $x$. Then $f$ is \textbf{at most the order of} $g$ as $x \to \infty$ if there is a positive integer $M$ for which

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### Example

$x = O(e^x)$ because $\frac{x}{e^x} \to 0$ as $x \to \infty$, i.e. we can select an $M$. 