Recurrence Relations

Book Problems

31. Solve the recurrence relation \( h_n = 4_n - 2 \) with initial values \( h_0 = 0 \) and \( h_1 = 1 \).

\[ h_n = 4_n - 2 \Rightarrow h_n - 4_n - 2 = 0 \]

The characteristic equation is

\[ x^n - 4x^{n-2} = 0 \Rightarrow x^2 - 4 = 0 \]

When we factor this, we see the roots are \( x = \pm 2 \). Therefore, all solutions are of the form

\[ h_n = \alpha_1 2^n + \alpha_2 (-2)^n \]

When we apply the initial conditions, we get

\[ h_0 = 0 : \alpha_1 + \alpha_2 = 0 \]
\[ h_1 = 1 : 2\alpha_1 - 2\alpha_2 = 1 \]

Solving this gives \( \alpha_1 = \frac{1}{4} \) and \( \alpha_2 = -\frac{1}{4} \), so

\[ h_n = \frac{1}{4} 2^n - \frac{1}{4} (-2)^n \]

We can simplify this to

\[ h_n = 2^{n-2} - (-2)^{n-2} \]

33. Solve the recurrence relation \( h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3} \); \( n \geq 3 \) with initial values \( h_0 = 0 \), \( h_1 = 1 \) and \( h_2 = 2 \).

\[ h_n = h_{n-1} + 9h_{n-2} - 9h_{n-3} = h_n - h_{n-1} - 9h_{n-2} + 9h_{n-3} = 0 \]

Our characteristic equation is

\[ x^n - x^{n-1} - 9x^{n-2} + 9x^{n-3} = 0 \Rightarrow x^3 - x^2 - 9x + 9 = 0 \]

When we factor, we see

\[ x^3 - x^2 - 9x + 9 = x^2(x - 1) - 9(x - 1) = (x^2 - 9)(x - 1) = 0 \]

So the characteristic roots are \( x = 1, \pm 3 \). Since the roots are distinct, we have that all solutions are of the form

\[ h_n = \alpha_1 (1)^n + \alpha_2 (3)^n + \alpha_3 (-3)^n \]

When we apply the initial conditions we get

\[ h_0 = 0 : \alpha_1 + \alpha_2 + \alpha_3 = 0 \]
\[ h_1 = 1 : \alpha_1 + 3\alpha_2 - 3\alpha_3 = 1 \]
\[ h_2 = 2 : \alpha_1 + 9\alpha_2 + 9\alpha_3 = 2 \]
Solving this gives $\alpha_1 = -\frac{1}{4}$, $\alpha_2 = \frac{1}{3}$ and $\alpha_3 = -\frac{1}{12}$ and so

$$h_n = \frac{1}{4}(1)^n + \frac{1}{3}3^n - \frac{1}{12}(-3)^n$$

And this can be simplified to

$$h_n = \frac{1}{4} + 3^{n-1} + \frac{1}{4}(-3)^{n+1}$$

34. Solve the recurrence relation $h_n = 8h_{n-1} - 16h_{n-2}$ ($n \geq 2$) with initial values $h_0 = -1$ and $h_1 = 0$.

Rewriting this gives our characteristic equation

$$x^n - 8x^{n-1} + 16x^{n-2} = 0 \Rightarrow x^2 - 8x + 16 = 0$$

Factoring gives $(x - 4)(x - 4) = (x - 4)^2 = 0$, so we get a double root at $x = 4$. This means that all solutions are of the form

$$h_n = \alpha_1 4^n + \alpha_2 n4^n$$

When we apply the initial conditions, we see

$$h_0 = -1 : \alpha_1 = -1$$

$$h_1 = 0 : 4\alpha_1 + 4\alpha_2 = 0$$

Solving this system of equations gives that $\alpha_1 = -1$ and $\alpha_2 = 1$. So, our solution is

$$h_n = -4^n + n4^n \Rightarrow h_n = 4^n(n - 1)$$

38. Solve the following recurrence relations by examining the first few values for a formula and the proving your conjectured formula by induction.

(a) $h_n = 3h_{n-1}$, ($n \geq 1$); $h_0 = 1$

$$h_1 = 3h_0 = 3$$

$$h_2 = 3h_1 = 3^2h_0 = 3^2$$

$$h_3 = 3h_2 = 3^2h_1 = 3^3h_0 = 3^3$$

Our conjecture is that $h_n = 3^n$.

Proof: (by induction on $n$)

For our base case, notice that $h_0 = 1 = 3^0$.

Assume $h_n = 3^n$. We want to show that $h_{n+1} = 3^{n+1}$.

Consider

$$h_{n+1} = 3h_n = 3(3^n)$$

by our inductive hypothesis. And finally,

$$h_{n+1} = 3(3^n) = 3^{n+1}$$

which is the desired expression.
(b) \( h_n = h_{n-1} - n + 3, (n \geq 1); \ h_0 = 2 \)

\[
\begin{align*}
    h_1 &= h_0 - 1 + 3 = 4 \\
    h_2 &= h_1 - 2 + 3 = 5 \\
    h_3 &= h_2 - 3 + 3 = 5 \\
    h_4 &= h_3 - 4 + 3 = 4 \\
    h_5 &= h_4 - 5 + 3 = 2 \\
    h_6 &= h_5 - 6 + 3 = -1
\end{align*}
\]

Our conjecture is that \( h_n = 2 + 3n - \sum_{k=0}^{n} k = 2 + 3n - \frac{n(n+1)}{2} \).

For our base case, notice that \( h_1 = 2 + 3^1 - 1 = 4 \).

Assume \( h_n = 3n - \sum_{k=0}^{n} k = 2 + 3n - \frac{n(n+1)}{2} \). We want to show that

\[
    h_{n+1} = 2 + 3(n + 1) - \frac{(n + 1)(n + 2)}{2}
\]

Consider

\[
    h_{n+1} = h_n - (n + 1) + 3
\]

\[
    = \left(2 + 3n + \sum_{k=0}^{n} k\right) - (n + 1) + 3
\]

by our inductive hypothesis. Then

\[
    \left(2 + 3n + \sum_{k=0}^{n} k\right) - (n + 1) + 3
\]

\[
    = 2 + (3n + 3) - \left(\sum_{k=0}^{n} k + (n + 1)\right)
\]

\[
    = 2 + 3(n + 1) - \sum_{k=0}^{n+1} k
\]

\[
    = 2 + 3(n + 1) - \frac{(n + 1)(n + 2)}{2}
\]

which is the desired result.

(c) \( h_n = -h_{n-1} + 1, (n \geq 1); \ h_0 = 0 \)

\[
\begin{align*}
    h_1 &= -h_0 + 1 = 1 \\
    h_2 &= -h_1 + 1 = 0 \\
    h_3 &= -h_2 + 1 = 1 \\
    h_4 &= -h_3 + 1 = 0
\end{align*}
\]
Our conjecture is that $h_n \frac{(-1)^{n+1}+1}{2}$.

For our base case, notice that $h_0 = \frac{(-1)^1+1}{2} = 0$.
Assume that $h_n = \frac{(-1)^{n+1}+1}{2}$. We want to show that $h_{n+1} = \frac{(-1)^{n+2}+1}{2}$.
Consider

$$h_{n+1} = -h_n + 1 = -\frac{(-1)^{n+1}+1}{2} + 1$$

by our inductive hypothesis. Now,

$$\begin{align*}
\frac{(-1)^{n+1}+1}{2} + 1 &= \frac{-(-1)^{n+1} - 1}{2} + \frac{2}{2} \\
&= \frac{(-1)^{n+2} - 1}{2} + \frac{2}{2} \\
&= \frac{(-1)^{n+2} + 1}{2}
\end{align*}$$

which was the desired result.

(d) $h_n = -h_{n-1} + 2, \; (n \geq 1); \; h_0 = 1$

$$h_1 = -h_0 + 1 = 1$$
$$h_2 = -h_1 + 2 = 1$$
$$h_3 = -h_2 + 2 = 1$$
$$h_4 = -h_3 + 2 = 1$$

Our conjecture is that $h_n = 1$.

For our base case, notice that $h_1 = -1 + 2 = 1$.
Assume $h_n = 1$. We want to show $h_{n+1} = 1$.
Consider

$$h_{n+1} = -h_n + 2$$
$$= -1 + 2$$
$$= 1$$

which is the desired result.

(e) $h_n = 2h_{n-1} + 1, \; (n \geq 1); \; h_0 = 1$

$$h_1 = 2h_0 + 1 = 3$$
$$h_2 = 2h_1 + 1 = 2(3) + 1 = 7$$
$$h_3 = 2h_2 + 1 = 2(7) + 1 = 15$$
Our conjecture is that \( h_n = 2^{n+1} - 1 \).

For our base case, notice that \( h_0 = 2^{0+1} - 1 = 0 \).
Assume \( h_n = 2^{n+1} - 1 \) and we want to show that \( h_{n+1} = 2^{n+2} - 1 \).
Consider
\[
    h_{n+1} = 2h_n + 1
\]
\[
    = 2(2^{n+1} - 1) + 1
\]
by our inductive hypothesis. Simplifying, we see
\[
    2(2^{n+1} - 1) + 1 = 2^{n+2} - 2 + 1 = 2^{n+2} - 1
\]
which is the desired result.

Additional Problems

1. Consider the recurrence \( a_n = 15a_{n-1} - 44a_{n-2} \). Show that each of the following sequences is a solution.

   (a) \( \{4^n\} \)

   \[
   4^n = 15 \cdot 4^{n-1} - 44 \cdot 4^{n-2}
   \]
   \[
   4^2 = 15 \cdot 4 - 44
   \]
   \[
   16 = 60 - 44
   \]
   \[
   16 = 16
   \]

   So, \( \{4^n\} \) is a solution.

   (b) \( \{3 \cdot 11^n\} \)

   \[
   3 \cdot 11^n = 15 \cdot 3 \cdot 11^{n-1} - 44 \cdot 3 \cdot 11^{n-2}
   \]
   \[
   11^2 = 15 \cdot 11 - 44
   \]
   \[
   121 = 165 - 44
   \]
   \[
   121 = 121
   \]

   So, \( \{3 \cdot 11^n\} \) is a solution.

   (c) \( \{4^n - 11^n\} \)

   \[
   4^n - 11^n = 15(4^{n-1} - 11^{n-1}) - 44(4^{n-2} - 11^{n-2})
   \]
   \[
   4^n - 11^n = 15 \cdot 4^{n-1} - 15 \cdot 11^{n-1} - 44 \cdot 4^{n-2} + 44 \cdot 11^{n-2}
   \]
   \[
   4^n - 15 \cdot 4^{n-1} + 44 \cdot 4^{n-2} = 11^n - 15 \cdot 11^{n-1} + 44 \cdot 11^{n-2}
   \]
   \[
   4^{n-2}(4^2 - 15 \cdot 4 + 44) = 11^{n-2}(11^2 - 15 \cdot 11 + 44)
   \]
   \[
   4^{n-2}(16 - 60 + 44) = 11^{n-2}(121 - 165 + 44)
   \]
   \[
   4^{n-2}(0) = 11^{n-2}(0)
   \]
   \[
   0 = 0
   \]

   So, \( \{4^n - 11^n\} \) is a solution.
(d) \( \{4^{n+1}\} \)

\[
\begin{align*}
4^{n+1} &= 15 \cdot 4^n - 44 \cdot 4^{n-1} \\
4^{n+1} - 15 \cdot 4^n + 44 \cdot 4^{n-1} &= 0 \\
4^2 - 15 \cdot 4 + 44 &= 0 \\
16 - 60 + 44 &= 0 \\
0 &= 0
\end{align*}
\]

So, \( \{4^{n+1}\} \) is a solution.