Problem Set 1 Solutions

Section 1.1

1. Which of the following sentences are statements?

(a) \(3^2 + 4^2 = 5^2\).
    Statement

(b) \(a^2 + b^2 = c^2\).
    Not a statement because we have variables. Depending on the choice of those variables, we could have a true or a false statement.

(c) There exists integers \(a\), \(b\), and \(c\) such that \(a^2 + b^2 = c^2\).
    Statement because we have conditions on the values of \(a\), \(b\) and \(c\).

(d) If \(x^2 = 4\) then \(x = 2\).
    Not a statement because we don’t have any quantification of \(x\).

(e) For each real number \(x\), if \(x^2 = 4\), then \(x = 2\).
    Statement.

(f) For each real number \(t\), \(\sin^2 t + \cos^2 t = 1\).
    Statement.

(g) \(\sin(x) < \sin(\frac{\pi}{4})\).
    Not a statement because we have no quantification of \(x\).

(h) If \(n\) is a prime number, then \(n^2\) has three positive factors.
    Statement.

(i) \(1 + \tan^2 \theta = \sec^2 \theta\).
    Not a statement because there is no quantification of \(\theta\).

(j) Every rectangle is a parallelogram.
    Statement.

(k) Every even natural number greater than or equal to 4 is the sum of two prime numbers.
    Statement.

5. Let \(P\) be the statement “Student X passed every assignment in Calculus I,” and let \(Q\) be the statement “Student X received a grade of C or better in Calculus I.”

(a) What does it mean for \(P\) to be true? What does it mean for \(Q\) to be true?
    If \(P\) is true then student \(X\) turned in all assignments in Calculus I. If \(Q\) is true then student \(X\) received a grade of C or better in Calculus I.

(b) Suppose that Student X passed every assignment in Calculus I and received a grade of B-, and that the instructor made the statement \(P \rightarrow Q\). Would you say that the instructor lied or told the truth?
    The instructor told the truth as \(P \rightarrow Q\) is true if both \(P\) and \(Q\) are true.

(c) Suppose that Student X passed every assignment in Calculus I and received a grade of C-, and that the instructor made the statement \(P \rightarrow Q\). Would you say that the instructor lied or told the truth?
    The instructor lied because if \(P\) is true and \(Q\) is false then \(P \rightarrow Q\) is false.
(d) Now suppose that Student X did not pass two assignments in Calculus I and received a grade of D, and that the instructor made the statement $P \rightarrow Q$. Would you say that the instructor lied or told the truth?

The instructor did not lie as the student did not turn in all assignments, the instructor did not have to give a grade of at least C to be truthful. This is equivalent to the truth of the statement $P \rightarrow Q$ when both are false.

(e) How are Parts (5b), (5c), and (5d) related to the truth table for $P \rightarrow Q$?

They are precisely the truths and falsehood for $P \rightarrow Q$ when both are true, when $P$ is true and $Q$ is false, and when both are false.

6. Following is a statement of a theorem which can be proven using calculus or precalculus mathematics. For this theorem, $a$, $b$, and $c$ are real numbers.

**Theorem** If $f$ is a quadratic function of the form $f(x) = ax^2 + bx + c$ and $a < 0$, then the function $f$ has a maximum value when $x = \frac{-b}{2a}$.

Using only this theorem, what can be concluded about the functions given by the following formulas?

(a) $g(x) = -8x^2 + 5x - 2$

This quadratic satisfies the required conditions, so we can conclude that the maximum value occurs at $x = \frac{5}{16}$.

(b) $h(x) = -\frac{1}{7}x^2 + 3x$

This quadratic satisfies the required conditions, so we can conclude that the maximum value occurs at $x = \frac{9}{2}$.

(c) $k(x) = 8x^2 - 5x + 7$

We know noting because $a > 0$.

(d) $j(x) = -\frac{71}{99}x^2 + 210$

This quadratic satisfies the required conditions, so we can conclude that the maximum value occurs at $x = 0$.

(e) $f(x) = -4x^2 - 3x + 7$

This quadratic satisfies the required conditions, so we can conclude that the maximum value occurs at $x = -\frac{3}{8}$.

(f) $F(x) = -x^4 + x^3 + 9$

We know nothing because this is not a quadratic function.

9.

(a) Is the set of natural numbers closed under division?

No as we can easily find two natural numbers whose quotient is a rational number.

(b) Is the set of rational numbers closed under division?

No because division by zero is not defined.

(c) Is the set of nonzero rational numbers closed under division?

Yes.
(d) Is the set of positive rational numbers closed under division?  
Yes.
(e) Is the set of positive real numbers closed under subtraction?  
No. We can easily find two real numbers $a > b$ such that $b - a < 0$.
(f) Is the set of negative rational numbers closed under division?  
No as the quotient of two negative rational numbers is a positive rational number.
(g) Is the set of negative integers closed under addition?  
Yes. The sum of two negative integers will always be negative.

Section 1.2

2. Construct a know-show table for each of the following statements and then write a formal proof for one of the statements.

(a) If $x$ is an even integer and $y$ is an even integer, then $x + y$ is an even integer.  
Assume $x$ and $y$ are even integers. We want to show that $x + y$ is also an even integer. Since $x$ and $y$ are even integers, there exist $k, m \in \mathbb{Z}$ such that $x = 2k$ and $y = 2m$. Consider  
$$x + y = 2k + 2m = 2(k + m)$$  
where $k + m$ is an integer by the closure of the integers under addition. Therefore, if $x$ and $y$ are even integers, so is $x + y$. Q.E.D.

(b) If $x$ is an even integer and $y$ is an odd integer, then $x + y$ is an odd integer.  
Assume that $x$ is an even integer and $y$ is an odd integer. We want to show that $x + y$ is an odd integer. If $x$ is an even integer, then there exists $k \in \mathbb{Z}$ such that $x = 2k$. Also, since $y$ is an odd integer, there is an $m \in \mathbb{Z}$ such that $y = 2m + 1$. Consider $$x + y = 2k + 2m + 1 = 2(k + m) + 1$$  
where $k + m$ is an integer by the closure of integers under addition. Therefore, the sum of $x$ and $y$ satisfies the necessary form for inclusion in the set of odd integers when $x$ is even and $y$ is odd. Q.E.D.

(c) If $x$ is an odd integer and $y$ is an odd integer, then $x + y$ is an even integer.  
Assume $x$ and $y$ are odd integers. We want to show that their sum is an even integer. Since $x$ and $y$ are odd integers, there exists $k, m \in \mathbb{Z}$ such that $x = 2k + 1$ and $y = 2m + 1$. Consider  
$$x + y = 2k + 1 + 2m + 1 = 2k + 2m + 2 = 2(k + m + 1)$$  
where $k + m + 1$ is an integer by the closure of integers under addition and so we have shown that the sum $x + y$ satisfies the correct form for inclusion in the set of even integers. Q.E.D.

3. Construct a know-show table for each of the following statements and then write a formal proof for one of the statements.
(a) If \( m \) is an even integer and \( n \) is an integer, then \( m \cdot n \) is an even integer.
Assume that \( m \) is an even integer and that \( n \) is any integer. We will show that \( m \cdot n \) is an even integer.
Since \( m \) is an even integer, there exists a \( k \in \mathbb{Z} \) such that \( m = 2k \). Consider
\[
m \cdot n = 2k \cdot n = 2(kn)
\]
where \( kn \) is an integer by the closure of the integers under multiplication. Therefore, \( m \cdot n \) satisfies the correct form for inclusion in the set of even integers whenever \( m \) is even and \( n \) is any arbitrary integer. Q.E.D.

(b) If \( n \) is an even integer, then \( n^2 \) is an even integer.
Assume \( n \) is an even integer. We will show that \( n^2 \) is also an even integer.
Since \( n \) is even, there exists \( k \in \mathbb{Z} \) such that \( n = 2k \). Consider
\[
n^2 = (2k)^2 = 4k^2 = 2(2k^2)
\]
where \( 2k^2 \) is an integer by the closure of the set of integers under multiplication. Therefore, since \( n \) is an even integer, we have shown that \( n^2 \) must be as well. Q.E.D.

(c) If \( n \) is an odd integer, then \( n^2 \) is an odd integer.
Assume \( n \) is an odd integer. We want to show that \( n^2 \) is an odd integer as well.
Since \( n \) is odd, there exists \( k \in \mathbb{Z} \) such that \( n = 2k + 1 \). Consider
\[
n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1
\]
where \( 2k^2 + 2k \) is an integer by the closure of the integers under addition and multiplication. So, if \( n \) is odd, then we have shown that \( n^2 \) is as well. Q.E.D.

7. Are the following statements true or false? Justify your conclusions.

1. If \( a, b \) and \( c \) are integers, then \( ab + ac \) is an even integer.
   This statement is false. If \( a \) and \( b \) are odd and \( c \) is even, then as we have previously shown, we would have the sum of an even integer and an odd integer, which is odd.

2. If \( b \) and \( c \) are odd integers and \( a \) is an integer, then \( ab + ac \) is an even integer.
   This is a true statement. Since \( b \) and \( c \) are odd, we need only consider whether \( a \) is even or odd. If \( a \) is even, then \( ab \) and \( ac \) are both even, as previously shown, and so their sum is even. If \( a \) is odd, then \( ab \) and \( ac \) are both odd integers, and we have previously shown that the sum of two odd integers is an even integer.

11. Let \( a, b, \) and \( c \) be real numbers with \( a \neq 0 \). The solutions of the quadratic equation \( ax^2 + bx + c = 0 \) are given by the quadratic formula, which states that the solutions are \( x_1 \) and \( x_2 \), where
\[
x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}
\]
(a) Prove that the sum of the two solutions of the quadratic equation \( ax^2 + bx + c = 0 \) is equal to \( -\frac{b}{a} \).
Let the two solutions to the quadratic equation \( ax^2 + bx + c = 0 \) be as stated above. We will show their sum is \(-\frac{b}{a}\). Consider

\[
\begin{align*}
x_1 + x_2 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-b + \sqrt{b^2 - 4ac} + -b - \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{2b}{2a} \\
&= \frac{b}{a}
\end{align*}
\]

So, if the solutions of the quadratic equation are as given above, then their sum is \(-\frac{b}{a}\).

(b) Prove that the product of the two solutions of the quadratic equation \( ax^2 + bx + c = 0 \) is equal to \( \frac{c}{a} \).

Assume the solutions to the quadratic equation \( ax^2 + bx + c = 0 \). We will show the product of these roots is \( \frac{c}{a} \). Consider

\[
\begin{align*}
x_1 \cdot x_2 &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{b^2 - (b^2 - 4ac)}{4a^2} \\
&= \frac{4ac}{4a^2} \\
&= \frac{c}{a}
\end{align*}
\]

So, if the solutions of the quadratic equation are as given above, then their product is \( \frac{c}{a} \).

Section 2.1

2. Suppose that \( P \) and \( Q \) are statements for which \( P \rightarrow Q \) is true and for which \( \neg Q \) is true. What conclusion (if any) can be made about the truth value of each of the following statements?

(a) \( P \)

Since \( \neg Q \) is true, then \( Q \) must be false. But since \( P \rightarrow Q \) is true, it must be the case that \( P \) is false.

(b) \( P \land Q \)

Since \( P \) is false and \( Q \) is false, \( P \land Q \) is false as well.

(c) \( P \lor Q \)

Both \( P \) and \( Q \) are false, so \( P \lor Q \) is false as well.

4. Suppose that \( P \) and \( Q \) are statements for which \( Q \) is false and \( \neg P \rightarrow Q \) is true (and it is not known if \( R \) is true or false). What conclusion (if any) can be made about the truth value of each of the following statements?
(a) \(\neg Q \rightarrow P\)
Since \(Q\) is false, it must be the case the \(\neg P\) is false in order for \(\neg P \rightarrow Q\) to be a true statement. Therefore, \(P\) is a true statement. Since \(P\) is true, for any value of \(\neg Q\), we would have that \(\neg Q \rightarrow P\) is true.

(b) \(P\)
As stated in (a), \(P\) is a true statement.

(c) \(P \land R\)
We know \(P\) is true, but since we are looking for \(P \land R\), we cannot determine the truth of this statement. If \(R\) is true, then \(P \land R\) is true but if \(R\) is false then \(P \land R\) is false.

(d) \(R \rightarrow \neg P\)
Since \(P\) is true, \(\neg P\) is false. Therefore \(R \rightarrow \neg P\) is inconclusive. We would have a true statement is \(R\) is false but a false statement is \(R\) is true.

6. Construct a truth table for each of the following statements:

(a) \(P \lor \neg Q\)

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(b) \(\neg (P \lor Q)\)

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(c) \(\neg P \lor \neg Q\)

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(d) \(\neg P \land \neg Q\)

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We get the same truth table for \( \neg(P \lor Q) \) and for \( \neg P \land \neg Q \).

9. Let \( P \) stand for “the integer \( x \) is even,” and let \( Q \) stand for “\( x^2 \) is even.” Express the conditional statement \( P \rightarrow Q \) in English using

(a) The “if then” form of the conditional statement
   If the integer \( x \) is even then \( x^2 \) is even.

(b) The word “implies”
   The integer \( x \) is even implies that \( x^2 \) is even.

(c) The “only if” form of the conditional statement
   The integer \( x \) is even only if \( x^2 \) is even.

(d) The phrase “is necessary for”
   \( x^2 \) is an even integer is necessary for \( x \) to be an even integer.

(e) The phrase “is sufficient for”
   \( x \) is an even integer is sufficient for \( x^2 \) to be even.

11. For statements \( P \) and \( Q \), use truth tables to determine if each of the following statements is a tautology, a contradiction, or neither.

(a) \( \neg Q \lor (P \rightarrow Q) \).

\[
\begin{array}{ccc}
P & Q & \neg Q & P \rightarrow Q & \neg Q \lor (P \rightarrow Q) \\
T & T & F & T & T \\
T & F & T & F & T \\
F & T & F & T & T \\
F & F & T & T & T \\
\end{array}
\]

So, \( \neg Q \lor (P \rightarrow Q) \) is a tautology.

(b) \( Q \land (P \land \neg Q) \).

\[
\begin{array}{ccc}
P & Q & \neg Q & P \land \neg Q & Q \land (P \land \neg Q) \\
T & T & F & F & F \\
T & F & T & T & F \\
F & T & F & F & F \\
F & F & T & F & F \\
\end{array}
\]

So, \( Q \land (P \land \neg Q) \) is a contradiction.

(c) \( (Q \land P) \land (P \rightarrow \neg Q) \).

\[
\begin{array}{ccc}
P & Q & Q \land P & \neg Q & P \rightarrow \neg Q & (Q \land P) \land (P \rightarrow \neg Q) \\
T & T & T & F & F & F \\
T & F & F & T & T & F \\
F & T & F & F & T & F \\
F & F & F & T & T & F \\
\end{array}
\]

So, \( (Q \land P) \land (P \rightarrow \neg Q) \) is a contradiction.
(d) \( \neg Q \rightarrow (P \land \neg P) \).

\[
\begin{array}{cc|c|c|c|c}
P & Q & \neg Q & \neg P & P \land \neg P & \neg Q \rightarrow (P \land \neg P) \\
\hline
T & T & F & F & F & T \\
T & F & T & F & F & F \\
F & T & F & T & T & T \\
F & F & T & T & F & F \\
\end{array}
\]

So, \( \neg Q \rightarrow (P \land \neg P) \) is neither a tautology or a contradiction.

14. **Working with Truth Values of Statements.** Suppose that \( P \) and \( Q \) are true statements, that \( U \) and \( V \) are false statements, and that \( W \) is a statement and it is not known if \( W \) is true or false.

Which of the following statements are true, which are false, and for which statements is it not possible to determine if it is true or false? Justify your conclusions.

(a) \( (P \lor Q) \lor (U \land W) \)

Since \( P \) and \( Q \) are true, \( P \lor Q \) is true and so this statement is true regardless of whether or not \( W \) is true.

(b) \( P \land (Q \rightarrow W) \)

This is indeterminant. Since \( W \) is indeterminant, we do not know whether \( Q \rightarrow Q \) is true or false because \( Q \) is known to be true. If \( Q \) was false then the value of \( W \) would not matter.

(c) \( P \land (W \rightarrow Q) \)

Since \( Q \) is true, \( W \rightarrow Q \) is a true statement regardless of whether or not \( W \) is true. Since \( P \) is known to be true, we have that \( P \land (W \rightarrow Q) \) is a true statement.

(d) \( W \rightarrow (P \land U) \)

\( P \land U \) is false, so this statement is indeterminant. If \( W \) is true then we have a false statement but if \( W \) is false then we have a true statement.

(e) \( W \rightarrow (P \land \neg U) \)

\( P \land \neg U \) is a true statement since \( P \) is true and \( U \) is known to be false, making \( \neg U \) a true statement. Since the conclusion of an implication is true, the hypothesis is irrelevant, so we have a true statement.

(f) \( (\neg P \lor \neg U) \land (Q \lor \neg V) \)

\( \neg P \lor \neg U \) is a true statement. \( P \) is true so \( \neg P \) is false, and \( U \) is false so \( \neg U \) is true. \( Q \lor \neg V \) is also true since both statements are true here. So, \( (\neg P \lor \neg U) \land (Q \lor \neg V) \) is a true statement.

(g) \( (P \land \neg V) \land (U \lor W) \)

This statement is indeterminant. \( P \lor \neg V \) is true because \( P \) is a true statement and the negation of \( V \) (false) is also true. So, we get our statement to be true if \( U \lor W \) is true and a false statement if \( U \lor W \) is false. Since \( U \) is false, the determination about \( U \lor W \) depends on whether \( W \) is true or false, and since this is unknown, we do not know whether or not our statement is true.
(h) \((P \lor \neg Q) \rightarrow (U \land W)\)

\(U \land W\) is false regardless of whether or not \(W\) is true because \(U\) is known to be false. Since \(P\) is true, \(P \lor \neg Q\) is true, so we have a true statement implying a false statement, which is false.

(i) \((P \lor W) \rightarrow (U \land W)\)

\(P \lor W\) is true regardless of whether \(W\) is true or false because \(P\) is true. \(U \land W\) is false regardless of whether \(W\) is true or false because \(U\) is false. So, we have a true statement implying a false statement which is false.

(j) \((U \land \neg V) \rightarrow (P \land W)\)

\(U \land \neg V\) is false because \(U\) is false and \(\neg V\) is true. Since the hypothesis is false, the conclusion is irrelevant; the statement \((U \land \neg V) \rightarrow (P \land W)\) is true since there is no possible way we can get true implying false when the hypothesis is true.