§ 6.3 Generalized Permutations and Combinations
Example
How many ways can I assign 3 tasks to people from a class of 10 students if people can be assigned more than one task?
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A Different Kind of Example

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A Different Kind of Example

Suppose the people are different slots and let ♣ indicate who a task was assigned to.
We can assign all three tasks to the same person.
A Different Kind of Example

We can assign the three tasks to three different people.
We can assign two tasks to one person and one to another.
Let’s remove the two end lines.
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Does order matter? i.e. Can you tell the difference between different dividers?
Since your answer is no, we are looking at some kind of combination.
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So we are looking to arrange \( n - 1 + r \) objects where order doesn’t matter.

So the idea is that we are randomly placing 12 objects without regard for which type is which, but we do need to take into account how many are of each type. So for our example, we would have

\[
_{12}C_3 = _{12}C_9
\]

ways to assign the job.
Generalized Combinations

Theorem

There are $C(n + r - 1, r) = C(n + r - 1, n - 1)$ \textit{r-combinations from a set with n elements when repetition of elements is allowed.}

Example

Suppose a cookie shop has 4 different kinds of cookies. In how many ways can 6 cookies be selected for purchase? Assume that only the type of cookie, and not the individual cookies or the order in which they are chosen, matters.

There are $9 \choose 6 = 84$ ways to select this half dozen.
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How many solutions does the equation

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have, where \(x_1\), \(x_2\) and \(x_3\) are nonnegative integers?
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How is this related to what we just did?

The solution is found by noting that we are looking for a way of selecting 11 items from a set with three elements so that $x_1$ items of type 1, $x_2$ item of type 2 and $x_3$ items of type 3 are chosen.
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\[ C(3 + 11 - 1, 11) = c(13, 11) = 78 \]
Example

How many words can we make with the letters in ‘SPRING’?

\[ \text{6!} = 720 \]

Example

How many words can we make with the letters in ‘SUMMER’?

It isn’t 720 ... Why not?

We have to factor in that there are two ‘Ms’ but that we can’t tell them apart.

\[ \text{6!} \div 2! = 360 \]
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Permutations Revisited

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How many words can we make with the letters of ‘SUCCESS’?
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We can first consider that we have 7 letters and we are going to choose where to place S’s. How many ways are there for us to do that?
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\[ 4C_2 \]
And now, we have two letters left, so let’s look at how many ways to arrange the U.
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$$2C_1$$
And now, we have two letters left, so let’s look at how many ways to arrange the U.

\[ 2C_1 \]

And finally, we are down to one letter.
And now, we have two letters left, so let’s look at how many ways to arrange the U.

$$2C_1$$

And finally, we are down to one letter.

$$1C_1$$
And now, we have two letters left, so let’s look at how many ways to arrange the U.

\[ 2C_1 \]

And finally, we are down to one letter.

\[ 1C_1 \]

Putting this all together, we have

\[ 7C_3 \cdot 4C_2 \cdot 2C_1 \cdot 1C_1 \]
Another Way For This Type

\[
7C_3 \cdot 4C_2 \cdot 2C_1 \cdot 1C_1 \\
= \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!}
\]
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\[
\begin{align*}
7C_3 \cdot 4C_2 \cdot 2C_1 \cdot 1C_1 &= \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} \\
&= \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} \\
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\end{align*}
\]
Another Way For This Type

\[ 7 \cdot \binom{3}{1} \cdot \binom{4}{2} \cdot \binom{1}{1} = \frac{7!}{3!4!} \cdot \frac{4!}{2!2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} = \frac{7!}{3!2!1!1!} = 420 \]
Another Way For This Type

\[
\begin{align*}
7C_3 \cdot 4C_2 \cdot 2C_1 \cdot 1C_1 &= \frac{7!}{3!4!} \cdot \frac{4!}{2!} \cdot \frac{2!}{1!1!} \cdot \frac{1!}{1!0!} \\
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&= \frac{7!}{3!2!1!1!} \\
&= 420
\end{align*}
\]

What are the numbers in the denominator?
Generalized Permutations

Theorem

The number of different permutations of \( n \) objects, where there are \( n_1 \) indistinguishable objects of type 1, \( n_2 \) indistinguishable objects of type 2, \ldots, and \( n_k \) indistinguishable objects of type \( k \), is

\[
\frac{n!}{n_1!n_2!\ldots n_k!}
\]
Generalized Permutations

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How many ways are there to deal 4 poker hands for 5 card draw from a standard deck of cards?
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Thoughts?
Generalized Permutations

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Thoughts?

\[
\frac{52!}{5!5!5!5!32!}
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Suppose 10 of you got together to work on discrete homework. For whatever reason, you decided to break into a group of 5, a group of 3 and a group of 2. In how many ways can you do this?
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\[ 10 \binom{5}{5} \cdot 5 \binom{3}{3} \cdot 2 \binom{2}{2} \]
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\[
\binom{10}{5} \cdot \binom{5}{3} \cdot \binom{2}{2}
= \frac{10!}{5! \cdot 5!} \cdot \frac{5!}{3! \cdot 2!} \cdot \frac{2!}{2! \cdot 0!}
\]
Example

Suppose 10 of you got together to work on discrete homework. For whatever reason, you decided to break into a group of 5, a group of 3 and a group of 2. In how many ways can you do this?

\[
\begin{align*}
\binom{10}{5} \cdot \binom{5}{3} \cdot \binom{2}{2} &= \frac{10!}{5!5!} \cdot \frac{5!}{3!2!} \cdot \frac{2!}{2!0!} \\
&= \frac{10!}{5!3!2!} \\
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\end{align*}
\]