1. Prove that isomorphic graphs have the same chromatic number and the same chromatic polynomial.

Let $G$ and $G'$ be isomorphic graphs. The, there is a function $\phi : G \rightarrow G'$ such that $\phi(u_i) = v_j$ for $u_i \in V(G)$ and $v_j \in V(G')$.

A way to consider this is using the Principle of Inclusion-Exclusion. Let’s consider an edge $e_i$ in $G$ that is incident to $u$ and $v$. Then $\phi(u) = u'$ and $\phi(v) = v'$ for $u', v' \in G'$, so since the isomorphism is preserved, we can label the edge incident to $u'$ and $v'$ as $e_i$ as well. Then, when considering property $a_i$ in the calculations of the chromatic polynomial, $N(e_i) = x$ precisely when $u$ and $v$ have the same color, which is precisely when $u'$ and $v'$ have the same color. When we repeat this process for all edges, we have all of the same values for $\sum N(a_i)$ when both graphs are considered. This trend will continue as $e_i$ and $e_j$ will be adjacent in $G$ precisely when $e_i$ and $e_j$ are adjacent in $G'$. When all possible edge grouping are considered, all sums $\sum N(a_i \ldots a_k)$ will be the same, resulting in the same chromatic polynomial.

2. Prove that the chromatic number of a disconnected graph is the largest chromatic number of its connected components.

Suppose $G$ is disconnected. We will assume $G$ has two components, as the same argument would hold for any finite number of components. So suppose the two components are $C_1$ and $C_2$ and that $\chi(C_2) \leq \chi(C_1) = k$. Since $C_1$ and $C_2$ are disconnected components of $G$, there are no edges between vertices of $C_1$ and $C_2$. So, apply the Greedy algorithm to $C_1$ to assign the $k$ colors. Then, when we move onto the second component, there are no vertices there adjacent to a vertex that has been colored, so we can assign to the first vertex the first color form our list. As we proceed with the second component, there are only vertices adjacent to one under consideration that are adjacent to vertices in $C_2$, so for the second vertex, we could use color 2, for the third vertex either color 1 or 2, etc. without consideration to the assignments in the first component. When we reach the last vertex of $C_2$, we assign the next color on the list that has not been assigned to a vertex adjacent to this vertex in $C_2$, but since $\chi(C_2) \leq \chi(C_1)$, we have used at most $k$ colors to properly color $C_2$.

5. Determine the chromatic number of the following graphs:
6. Prove that a graph with chromatic number equal to $k$ has at least $\binom{k}{2}$ edges.

By Brook’s Theorem, $\chi(G) \leq \Delta(G)$ for $G$ not complete or an odd cycle.

If $G$ is an odd cycle, then $\chi(C_{2n+1}) = 3$ for $n \geq 1$ and any odd cycle will have at least $\binom{3}{2} = 3$ edges.

If $G$ is the complete graph on $n$ vertices, then $\chi(K_n) = n$ and $\binom{n}{2}$ is the number of edges in a complete graph.

For all other graphs, $\binom{k}{2}$ is the number of edges that the largest complete subgraph would necessarily have, and the chromatic number for this subgraph is $k$. If $G$ contains other vertices, each is not adjacent to all of the vertices of the complete subgraph, so it can be placed in the same color class as one of the vertices that it is not adjacent to from this subgraph. So, in other words, the chromatic number of a graph is equal to that of the largest complete subgraph of the graph.

9. Let $G$ be a graph of order $n$ whose chromatic polynomial is $P_G(k) = k(k - 1)^{n-1}$ (i.e. the chromatic polynomial of $G$ is the same as that of a tree of order $n$). Prove that $G$ is a tree.

If a graph has a chromatic polynomial of the form $P_G(k) = k(k - 1)^{n-1}$, then in the expansion, the coefficient of the $k^{n-1}$ term is $n - 1$. So, we have a graph on $n$ vertices with $n - 1$ edges. It remains to be shown that the graph is connected.

Assume $G$ has $n$ components but has $P_G(k) = k(k - 1)^{n-1}$ for its chromatic polynomial. This is impossible, since by a generalization of the Two-Piece Theorem, we know $P_G(k) = \Pi P_{G_i}(k)$ where the components of $G$ are $G_i$, for $i = 1, \ldots, n$ and for each component, there would be a factor of $x$. That is, $P_G(k) = k^n P'_G(k)$, where $P'_G(k)$ is the rest of the polynomial with the term $k^n$ removed. Since the exponent for this term in $P_G(k) = k(k - 1)^{n-1}$ is one, we have only one component, and therefore we have a connected graph on $n$ vertices and $n - 1$ edge. So, the graph with this chromatic polynomial is a tree.

10. What is the chromatic number of a graph obtained from $K_n$ by removing one edge?

The chromatic number would be $n - 1$. For a complete graph on $n$ vertices, we know the chromatic number is $n$. If one edge is removed, we now have a pair of vertices that are no longer adjacent. So, they can be colored using the same color. The graph still has a complete number of edges.
subgraph on \( n - 1 \) vertices, so we would still need those \( n - 1 \) colors, but the nonadjacent vertices no longer need separate colors.

12. What is the chromatic number of a graph obtained from \( K_n \) by removing two edges with a common vertex?

Let \( G' \) be the graph \( K_n \) with the two adjacent edges removed. \( G' \) is guaranteed to still have a complete subgraph on \( n - 1 \) vertices, so the minimum chromatic number would be \( n - 1 \). And, by Brook’s Theorem, since \( G' \) is not a complete graph nor an odd cycle, the maximum chromatic number is \( n - 1 = \Delta(G') \). So, \( \chi(G') = n - 1 \).

13. What is the chromatic number of a graph obtained from \( K_n \) by removing two edges without a common vertex?

By Brooks’ Theorem, we know our chromatic number has decreased, since for a noncomplete graph \( G' \), \( \chi(G') \leq \Delta(G') \), where \( G' \) is the graph \( K_n \) with two nonadjacent edges removed. By removing these two nonadjacent edges, however, we don’t reach \( \Delta(G') \) as the edges being removed results in a largest subgraph of \( K_n - 2 \). This gives that \( \chi(G') = n - 2 \).

14. Prove that the chromatic polynomial of a cycle graph \( C_n \) equals \( (k - 1)^n + (-1)^n (k - 1) \).

This proof will be by induction on \( n \).

Our base case will be for \( C_3 \), so for \( n = 3 \). \( P_{C_3}(x) = x(x - 1)(x - 2) = (x - 1)(x^2 - x) = (x - 1)^3 + (-1)^3(x - 1) \).

Our induction hypothesis is that \( P_{C_n}(x) = (x - 1)^n + (-1)^n(x - 1) \). We want to show that \( P_{C_n+1}(x) = (x - 1)^{n+1} + (-1)^{n+1}(x - 1) \).

Consider the graph of \( C_{n+1} \). Apply the Fundamental Reduction Theorem to this graph, leaving us a path on \( n + 1 \) vertices when the edge is deleted and \( C_n \) for the contraction. This gives

\[
P_{C_{n+1}}(x) = P_{P_{n+1}}(x) - P_{C_n}(x)
\]

Now,

\[
P_{P_{n+1}}(x) = x(x - 1)^n
\]

and

\[
P_{C_n}(x) = (x - 1)^n + (-1)^n(x - 1)
\]

by our inductive hypothesis. Putting this all together, we have

\[
P_{C_{n+1}}(x) = x(x - 1)^n - [(x - 1)^n + (-1)^n(x - 1)] = (x - 1)(x - 1)^n - (-1)^n(x - 1) = (x - 1)^{n+1} + (-1)^{n+1}(x - 1)
\]

as required.

16. Prove that the polynomial \( k^4 - 4k^3 + 3k^2 \) is not the chromatic polynomial of any graph.

If we factor this polynomial, we get \( k^2(k - 1)(k - 2) \). There are two vertices with \( k \) choices of colors, so the graph necessarily has two components. But, the absolute value of the coefficient of the \( x^3 \) term is 4, so the graph must have 4 edges. Since the order of the graph is 4 and
there must be two components, the most edges we could have in a simple graph would be 3, and that would result in an isolated vertex. The only other option would be two vertices in each component (which wouldn’t make sense with this chromatic polynomial) and then the graph would only have two edges. So, it is impossible to have a connected graph on 4 vertices with this chromatic polynomial, and there are too many edges for there to be a disconnected graph with this chromatic polynomial.

17. Use Theorem 12.1.10 to determine the chromatic number of the following graph:

\[
\chi(G) = 4
\]

Additional Problems

1. If \( \overline{G} \) is the complement of \( G \), use induction to show that \( \chi(G) + \chi(\overline{G}) \leq n + 1 \).

   For our base case, consider the graph \( G \cup \overline{G} = K_2 \). There is only one edge, so if \( G = K_2 \), then \( \overline{G} \) is the empty graph on two vertices. \( \chi(G) = 2 \) and \( \chi(\overline{G}) = 1 \), satisfying the inequality.

   As our inductive hypothesis, suppose \( \chi(G) + \chi(\overline{G}) \leq n \) for a graph \( G \) of order \( n - 1 \). We want to show that \( \chi(G) + \chi(\overline{G}) \leq n + 1 \) for a graph of order \( n \).

   Consider a graph \( G \) with order \( n \). By our inductive hypothesis, we can color \( G - v \) and \( \overline{G} - v \), for any vertex \( v \), with at most \( n \) colors. Now, \( v \) is incident to \( n - 1 \) edges in \( G \) and \( \overline{G} \) together, so there is one of the \( n \) color classes that \( v \) is not adjacent to in \( G \) or \( \overline{G} \) and can be added to that color class. In the other graph, \( v \) may require an additional color, for a total of \( n + 1 \) as required.

2. Determine the chromatic polynomial for the following graphs:

   (a)

   \[
P_G(x) = x(x - 1)(x - 2)(x - 3) = x^4 - 6x^3 + 11x^2 - 6x
   \]
3. Find the graph corresponding to the given map. Then find $\chi(G)$ for the graph you create.
4. The following committees need to have meetings scheduled:

A = {Smith, Jones, Brown, Green}
B = {Jones, Wagner, Chase}
C = {Harris, Oliver}
D = {Harris, Jones, Mason}
E = {Oliver, Cummings, Larson}

Are three meeting times sufficient to schedule to committees so that no member has to be at two meetings simultaneously?
The committees will be the vertices and we will place an edge between committees when they share a member.

Since $\chi(G) = 3$, we can schedule the meetings so there are no conflicts using only three meeting times.

5. The following tours of garbage trucks in Boston are being considered (behind Mayor Menino’s back).

- Tour 1: The Prudential, the Garden, and the Charlestown Shipyard
- Tour 2: Back Bay, the Charlestown Shipyard, the Prudential, and the Museum of Fine Arts
• Tour 3: Symphony Hall, the Old North Church, and MIT
• Tour 4: Quincy Market and the Charlestown Shipyard
• Tour 5: Quincy Market, Fanueil Hall and the Prudential
• Tour 6: Symphony Hall, Fenway Park and the Old North Church
• Tour 7: Fanueil Hall, Harvard and the Old North Church

Assuming the sanitation workers refuse to work more than three days a week, can these tours be partitioned so that no site is visited more than once on a given day?

To solve, we will assign the vertices to be the tours and an edge will be placed when there are tours that have a common location.

\[
\begin{align*}
\text{Tour 1} & \quad \text{Tour 2} \\
\text{Tour 7} & \quad \text{Tour 3} \\
\text{Tour 6} & \quad \text{Tour 4} \\
\text{Tour 5} & 
\end{align*}
\]

What we would need is a 3-coloring, since the workers would only work 3 days a week. We have a complete subgraph on 4 vertices, however, so it must be that \(\chi(G) = 4\) and so we cannot do all tours in three days while visiting each location at most once per day.