Chapter 11 Homework

2. Determine each of the 11 nonisomorphic graphs of order 4, and give a planar representation of each.

3. Does there exist a graph of order 5 whose degree sequence equals \((4, 4, 3, 2, 2)\)?
   No. When we remove a vertex and consider the resulting subgraph, we get \((4, 4, 3, 2, 2) \rightarrow (3, 2, 1, 1)\), which has an odd number of odd vertices.

4. Does there exist a graph of order 5 whose degree sequence equals \((4, 4, 4, 2, 2)\)? a multigraph?
   No simple graph exists. When we consider the degree sequence and the removal of vertices, we would have \((4, 4, 4, 2, 2) \rightarrow (3, 3, 1, 1) \rightarrow (2, 0, 0)\). This would leave a vertex of degree 2, but no vertices for it to be adjacent to.
   We could have a multigraph, however. If we suppose that each vertex of degree 4 had a loop, then the removal of those loops would leave us with a degree sequence \((2, 2, 2, 2, 2)\), which is the sequence for \(C_5\).

12. Determine which pairs of the graphs in Figure 11.40 are isomorphic, and for those that are isomorphic, find an isomorphism.
$G_1$ and $G_2$ are isomorphic. The mapping is given by $\phi : G_1 \rightarrow G_2$ such that

\[
\begin{align*}
\phi(a) &= j', \\
\phi(b) &= c', \\
\phi(c) &= d', \\
\phi(d) &= e', \\
\phi(e) &= f', \\
\phi(f) &= i', \\
\phi(g) &= b', \\
\phi(h) &= h', \\
\phi(i) &= g'.
\end{align*}
\]

$G_3$ is not isomorphic to $G_1$, and since $G_1$ is isomorphic to $G_2$, then $G_3$ cannot be isomorphic to $G_2$ either.

13. Prove that, if two vertices of a general graph are joined by a walk, then they are joined by a path.

Let $u$ and $v$ be arbitrary vertices of a general graph $G$. Let a $u - v$ walk in $G$ be $u = v_0, v_1, \ldots, v_n = v$. If all $v_i$ are distinct for $i = 0, \ldots, n$ then the walk is a path. So suppose that there are repeated vertices on this $u - v$ walk. Beginning at $u$, let $v_k$ be the first vertex that occurs twice on the walk. Then, all vertices and edges between the two occurrences of $v_k$ can be deleted along with one of the $v_k$'s. The resulting walk will be a shorter $u - v$ walk as the only part that was removed was the closed cycle beginning at $v_k$. Repeat this for any other vertices that occur more than once on the walk and the final $u - v$ walk will have no repeated vertices, whereby making it a $u - v$ path.

15. Let $x$ and $y$ be vertices of a general graph, and suppose that there is a closed trail containing both $x$ and $y$. Must there be a cycle containing both $x$ and $y$?

No. Consider the graph

\[
\begin{align*}
x &\quad \quad \quad v \\
&\quad \quad \quad y
\end{align*}
\]

This graph contains the closed trail $v - x - v - y - v$. But, because vertices are repeated, this is not a cycle.
19. Let $G$ be a general graph and let $G'$ be the graph obtained from $G$ by deleting all loops and all but one copy of each edge with multiplicity greater than 1. Prove that $G$ is connected if and only if $G'$ is connected. Also prove that $G$ is planar if and only if $G'$ is planar.

Proof: (Connectivity) If a graph $G'$ is connected, adding edges cannot cause a graph to be disconnected. On the other hand, if a graph $G$ is connected and loops are removed, all edges incident to the vertex and a unique vertex will still exist, keeping the graph connected. Further, if there exists at least one edge between any two adjacent vertices, any repeated edges can be removed. The degree of the affected vertices will be decreased, but there will still be one edge between said vertices, keeping the graph connected.]

Proof: (Planarity) If $G$ is planar, then removing edges will not cause the graph to lose this property. If $G'$ is planar, then any copies of an existing edge will occupy the same region as the current edge and will not need to be drawn by crossing any existing edges. Also, loops can be drawn to be entirely within any region with the vertex as a 'corner', so no edges will need to be crossed to add a loop to a graph.

20. Prove that a graph of order $n$ with at least $\frac{(n-1)(n-2)}{2} + 1$ edges must be connected. Give an example of a disconnected graph of order $n$ with one fewer edge.

If we are required only $\frac{(n-1)(n-2)}{2}$ edges, then we would have the complete subgraph $K_{n-1}$ and one isolated vertex.

Suppose we have a graph of order $n$ that has two components. Suppose one is on $k$ vertices and the other is on the remaining $n-k$ vertices. Note that $1 \leq k \leq n-1$, otherwise we would have only one connected component. Then, the components could have a maximum of $\frac{k(k-1)}{2}$ and $\frac{(n-k)(n-k-1)}{2}$, respectively.

We want to show that if this graph is disconnected, it contains at most $\frac{(n-1)(n-2)}{2}$ edges. We have

$$\frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \leq \frac{(n-1)(n-2)}{2}$$

$$\Rightarrow 2k^2 + n^2 - 2nk - n \leq n^2 - 3n + 2$$

$$\Rightarrow 2k^2 - 2nk \leq -2n + 2$$

$$\Rightarrow k^2 - 1 \leq nk - n$$

$$\Rightarrow k + 1 \leq n$$

which is true for any choice of $k$, $1 \leq k \leq n-1$. So, if $G$ is disconnected then it has at least one less edge than we require. But, each component was complete, so the addition of any additional edge would necessarily make a vertex from each component adjacent.
21. Let $G$ be a general graph with exactly two vertices $x$ and $y$ of odd degree. Let $G^*$ be the general graph obtained by putting a new edge $\{x, y\}$ joining $x$ and $y$. Prove that $G$ is connected if and only if $G^*$ is connected.

Suppose we have a connected graph $G$. The addition of an edge cannot destroy this property, so $G^*$ must also be connected. On the other hand, consider the graph $G^*$. Since $x$ and $y$ are the only odd vertices of $G$, they must be in the same component, or the degree sum in two components would be odd, which is impossible. So, the addition of the edge incident to $x$ and $y$ would not change the connectivity of the graph since the two vertices were already in the same component, so $G$ is connected when $G^*$ is connected.

27. Determine the adjacency matrices of the first and second graphs in Figure 11.40.

$$I_1 = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}$$

$$I_2 = \begin{bmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{bmatrix}$$

29. Determine if the multigraphs in Figure 11.41 have Eulerian trails (closed or open). In case there is an Eulerian trail, use the algorithms presented in this chapter to construct one.

$G_1$ does not have an Eulerian trail because the graph contains 4 odd vertices.

$G_2$ does contain an Eulerian trail because all vertices are even. One Eulerian trail could be

$$h - a - b - c - d - e - f - g - h - i - a - b - i - j - b - k - c - d - k - f - j - k - e - f - i - g - h$$

30. Which complete graphs $K_n$ have closed Eulerian trails? open Eulerian trails?

Complete graphs with odd $n$, $n \geq 3$ will have closed Eulerian trails since all vertices will be even. Only $K_2$ will have an open Eulerian trail.
34. Determine all non isomorphic graphs of order at most 6 that have a closed Eulerian trail.

To solve, we will make two assumptions - that the graph is simple and that the graph is connected.

37. Solve the Chinese postman problem for the complete graph $K_6$.

Since $K_6$ is 5-regular, the graph does not contain an Eulerian circuit. So some repetition of edges is necessary to solve this problem.

A solution requiring the repetition of only 3 edges is

$$a - b - c - a - d - e - f - d - b - f - c - e - a - f - d - c - e - b - a$$

39. Call a graph cubic if each vertex has degree equal to 3. The complete graph $K_4$ is the smallest example of a cubic graph. Find an example of a connected, cubic graph that does not have a Hamilton path.
Using the hint, we would get something that look like

There are a couple of famous examples of cubic, non-Hamiltonian graphs.

Figure 1: Tutte Graph

Figure 2: Horton Graph

40. Let $G$ be a graph of order $n$ having at least

$$\frac{(n - 1)(n - 2)}{2} + 2$$
edges. Prove that $G$ has a Hamilton cycle. Exhibit a graph of order $n$ with one fewer edge that does not have a Hamilton cycle.

We can prove this by induction on $n$, the order of the graph.

For our base case, consider the graph on $n = 3$ vertices. This graph would require $\frac{(3-1)(3-2)}{2} + 2 = 3$ edges to be Hamiltonian.

Now, assume that a graph on $n - 1$ vertices with $\frac{(n-2)(n-3)}{2} + 2$ edges is Hamiltonian. We want to show that a graph on $n$ vertices with $\frac{(n-1)(n-2)}{2} + 2$ edges is Hamiltonian.

We know that the maximum degree of the graph must be at least as big as the average degree, and will equal the average if we have a regular graph. We have

$$\Delta(G) \geq 2 \left( \frac{(n-1)(n-2)}{2} + 2 \right) = \frac{(n-1)(n-2) + 4}{n} = \frac{n^2 - 4n + 6}{n} = n - 3 + \frac{6}{n}$$

That is, $\Delta(G) > n - 3$. Let $v$ be the vertex with maximum degree. There are two cases to consider.

Case 1: $\deg(v) = n - 2$.

Here, we have

$$|E(G - v)| \geq \frac{(n-1)(n-2)}{2} + 2 - (n-2)$$

$$= \frac{(n-1)(n-2) + 4 - 2(n-2)}{2}$$

$$= \frac{n^2 - 5n + 6 + 4}{2}$$

$$= \frac{(n-2)(n-3)}{2} + 2$$

So, by the inductive hypothesis, $G - v$ has a Hamiltonian cycle. Since $\deg(v) = n - 2$, we can find two adjacent vertices in $G$ that are adjacent to $v$ that are on this cycle in $G - v$, so we can add $v$ to the cycle in between these vertices to complete our Hamiltonian cycle in $G$.

Case 2: $\deg(v) = n - 1$

In this case, the graph $G - v$ does not satisfy the inductive hypothesis as we would have $\frac{(n-2)(n-3)}{2} + 1$ edges. $G - v$ is not a complete graph in $n - 1$ vertices, for if it was then the hypothesis would be satisfied. So for any nonadjacent vertices $x$ and $y$, add the edge $xy$ to $G - v$ and call this new graph $G'$. $G'$ satisfies the inductive hypothesis. If the Hamiltonian cycle in $G'$ does not contain the edge $xy$, we can add the vertex $v$ and the incident edges as we did in the first case. If $xy$ is on the Hamiltonian cycle, the the deletion of $xy$ in $H$ means that $G - v$ contains a Hamiltonian path that has as it’s endpoints $x$ and $y$. Since $\deg(v) = n - 1$, it necessarily is adjacent to $x$ and $y$, so this Hamiltonian path of $G - v$ taken with the edges $vx vy$ completes the Hamiltonian cycle in $G$.

An example of a graph with $\frac{(n-1)(n-2)}{2} + 1$ edges that does not have a Hamiltonian cycle would be a graph with a complete subgraph on $n - 1$ vertices and one other vertex that is only adjacent to one of the vertices of the clique. The graph could not be Hamiltonian since the vertex of degree one must have it’s one edge on any path containing that vertex, and there can be no Hamiltonian cycle in any graph that contains a bridge.
42. Prove Theorem 11.3.4.

A graph of order \( n \), in which the sum of the degrees of each pair of nonadjacent vertices is at least \( n - 1 \), has a Hamiltonian path.

Suppose we have a graph \( G \) has the property that any two nonadjacent vertices have a degree sum of at least \( n \). Then \( G \) is Hamiltonian. If, for two vertices \( u \) and \( v \) on this cycle, we remove the edge \( uv \), the resulting graph \( G' = G - uv \) contains a Hamiltonian path, but we are no longer guaranteed that it contains a Hamiltonian cycle. The degree of all other vertices remain unchanged, but the degree of both \( u \) and \( v \) is decreased by 1. So, since in \( G \) we have \( \deg(u) + \deg(v_i) \geq n \) and \( \deg(v) + \deg(v_i) \geq n \) for all \( v_i \) nonadjacent to \( u \) and \( v \), respectively, in \( G' \) we have \( \deg(u) + \deg(v_i) \geq n - 1 \) and \( \deg(v) + \deg(v_i) \geq n - 1 \).

44. Which complete bipartite graphs \( K_{m,n} \) have Hamiltonian cycles? Which have Hamiltonian paths?

If we have a bipartite graph \( K_{m,n} \), the graph will have a Hamiltonian cycle iff \( m = n \). The graph will have a Hamiltonian path, therefore, if \( m = n \) as well. In addition, the graph will have a Hamiltonian path if \( m \) and \( n \) differ by 1.

46. Prove that \( K_{m,n} \) is isomorphic to \( K_{n,m} \).

For \( K_{m,n} \), let the bipartition \( (X, Y) \) be such that \( |X| = m \) and \( |Y| = n \). For \( K_{n,m} \), let the bipartition \( (X', Y') \) be such that \( |X'| = n \) and \( |Y'| = m \). Note that each vertex in \( X \) is of degree \( n \) and each vertex in \( Y' \) is of degree \( n \) as well. Similarly, the degree of each vertex in \( Y \) is \( m \), as is the degree of each vertex in \( X' \). So, there exists an injection \( \phi : (X, Y) \rightarrow (X', Y') \) such that \( \phi(X) = Y' \) and \( \phi(Y) = X' \). That is, for \( x_i \in X \) and \( y'_i \in Y' \), \( \phi(x_i) = y'_i \) for \( i = 1, \ldots, n \), and for \( y_j \in Y \) and \( x'_j \in X' \), \( \phi(y_j) = x'_j \) for \( j = 1, \ldots, m \).

49. Let \( V = \{1, 2, \ldots, 20\} \) be the set of the first 20 positive integers. Consider the graphs whose vertex set is \( V \) and whose edge sets are defined below. For each graph, investigate whether the graphs \( (i) \) is connected (if not connected, determine the connected components), \( (ii) \) is bipartite, \( (iii) \) has an Eulerian trail, and \( (iv) \) has a Hamilton path.

(a) \( \{a, b\} \) is an edge if and only if \( a + b \) is even.
(b) \( \{a, b\} \) is an edge if and only if \( a + b \) is odd.
(c) \( \{a, b\} \) is an edge if and only if \( a \times b \) is even.
(d) \( \{a, b\} \) is an edge if and only if \( a \times b \) is odd.
(e) \( \{a, b\} \) is an edge if and only if \( a \times b \) is a perfect square.
(f) \( \{a, b\} \) is an edge if and only if \( a - b \) is divisible by 3.

(a) \( \{a, b\} \) is an edge if and only if \( a + b \) is even.

For \( \{a, b\} \) to be an edge, either \( a \) and \( b \) are both even or both odd.

(i) No, and each component was a copy of \( K_{10} \)
(ii) No, since each component is a clique (complete subgraph) and there is no way to partition a complete graph into a bipartition
(iii) No, since the graph is disconnected
(iv) No, since the graph is disconnected

(b) \(\{a, b\}\) is an edge if and only if \(a + b\) is odd.
Here, we get the edge \(\{a, b\}\) when one is even and the other is odd. So, each there is a
walk of length one from every vertex of opposite parity and a walk of length two from
every vertex of the same parity. Thus, the graph would be \(K_{10,10}\).

(i) Yes
(ii) Yes, \(K_{10,10}\)
(iii) Yes, since \(n\) is even and so \(deg(v_i) = 10 \forall v_i\)
(iv) Yes, since \(|X| = |Y|\)

(c) \(\{a, b\}\) is an edge if and only if \(a \times b\) is even.
For edge \(\{a, b\}\) to exist, either \(a\) or \(b\) or \(a\) and \(b\) must be even. This, the only vertices
that are not adjacent are those where both \(a\) and \(b\) are odd.

(i) Yes, as even vertices are all adjacent to each other and to vertices that are odd.
There exists a walk of length 2 between all odd vertices as all odd vertices are each
adjacent to all even vertices.
(ii) No, as the even vertices alone form a clique of order 10
(iii) No, as all even vertices are adjacent to every other vertex and so they have degree 19
(iv) All odd vertices have degree 10 as they are all adjacent to all of the even vertices.
Since they are the only nonadjacent vertices in the graph and for any two of them,
\(deg(a) + deg(b) = 20 = n\), by Ore’s Theorem the graph is Hamiltonian

(d) \(\{a, b\}\) is an edge if and only if \(a \times b\) is odd.
For \(\{a, b\}\) to be an edge, but \(a\) and \(b\) must be odd. This means that no even vertex has
any incident edges.

(i) No, since even vertices all have degree 0
(ii) No, since even vertices all have degree 0 and therefore are not adjacent to any vertex
in another subset of any partition
(iii) No, since \(G\) is disconnected
(iv) No, since \(G\) is disconnected

(e) \(\{a, b\}\) is an edge if and only if \(a \times b\) is a perfect square.
The only edges would be \(\{1, 4\}, \{1, 9\}, \{1, 16\}\) and \(\{2, 8\}\)

(i) No, as there are only 4 edges but 20 vertices
(ii) No, because there are vertices that have degree 0, so they will not be adjacent to
any vertex in any set of a partition
(iii) No, since \(G\) is disconnected
(iv) No, since \(G\) is disconnected

(f) \(\{a, b\}\) is an edge if and only if \(a - b\) is divisible by 3.
The vertices can be partitioned into three disjoint sets;
\(v \in X\) when \(v \equiv 0\)(mod3)
\(v \in Y\) when \(v \equiv 1\)(mod3)
\(v \in Z\) when \(v \equiv 2\)(mod3)
(i) No, since no edge can exist between vertices from different sets
(ii) No, since each vertex is adjacent to every vertex in its own set and not adjacent to all vertices in the other sets
(iii) No, since $G$ is disconnected
(iv) No, since $G$ is disconnected

50. What is the smallest number of edges that can be removed from $K_5$ to leave a bipartite graph?
$K_5$ has 10 edges. Our choices for a bipartite graph would either be with $|X| = 2$ and $|Y| = 3$ $|X| = 1$ and $|Y| = 4$. Note that choice of $X$ and $Y$ are arbitrary. Also, the complete bipartite graph, regardless of which we use, would have more edges than any other bipartite graph with the same partition.

If we were to use $K_{1,4}$ as our choice, we would have only 4 edges, so we would need to remove 6. But if we were to use $K_{2,3}$, we would need 6 edges and therefore would only remove 4. So, the answer here is that the smallest number of edges that can be removed to result in a bipartite graph would be 4.

51. Find a knight’s tour on the boards of the following sizes:
   (a) 5-by-5
   (b) 6-by-6
   (c) 7-by-7

53. Prove that a graph is a tree if and only if it does not contain any cycles, but the insertion of any new edge always creates exactly one cycle.

Suppose a graph $G$ is a tree. Then, for any two vertices, there is a unique path. Suppose $u$ and $v$ are nonadjacent vertices and the path $P_1$ is the unique $u - v$ path. Then, if we add the edge $uv$ to $G$, we have the cycle created by traversing $P_1$ from $u$ to $v$ and then return to $u$ using this newly added edge. Since there was a unique path from $u$ to $v$, this is the only cycle that could be created in $G + uv$. For suppose that a second cycle is created by the addition of the edge $uv$. Then the removal of this edge would leave a $u - v$ path $P_2$ that is distinct from $P_1$. Then, in $G$, we could traverse $P_1$ from $u$ to $v$ and then return from $v$ to $u$ via $P_2$, which creates a cycle and contradicts our assumption that $G$ contained no cycles.

On the other hand, consider a graph $G$ that contains no cycles but when any new edge is added, then the new graph does contain exactly one cycle. We must show that $G$ is connected. Let $u$ and $v$ be nonadjacent vertices of $G$. Then, $G + uv$ contains contains a cycle where $u$ and $v$ necessarily are on this cycle. Them the removal of this edge leaves us with a $u - v$
path. Since $u$ and $v$ are arbitrary nonadjacent vertices, $G$ is a connected, acyclic graph and is therefore a tree.

54. Which trees have an Eulerian path?

Only trees that are paths will have an Eulerian path. Here, there would be only two pendent vertices, and so only two vertices of odd degree.

55. Which trees have a Hamiltonian path?

Only when the tree is a path will it have a Hamiltonian path. That is, only when there are only two pendent vertices which serve as the initial and terminal points of the Hamiltonian path. If there was another pendent vertex, which there would have to be if the tree was not a path, then when we visit that vertex, we would not be able to leave the vertex to continue onto the terminal vertex.

56. Grow all the nonisomorphic trees of order 7.
57. Let \((d_1, d_2, \ldots, d_n)\) be a sequence of integers.

(a) Prove that there is a tree of order \(n\) with this degree sequence if and only if \(d_1, d_2, \ldots, d_n\) are positive integers with sum \(d_1 + d_2 + \ldots + d_n = 2(n - 1)\).

Suppose we have a connected graph with degree sum \(2(n - 1)\). Since the graph is connected, all \(d_i > 0\). A degree sum of \(2(n - 1)\) is for that of a graph with order \(n\) and \(n - 1\) edges, and we have previously shown that a connected graph with \(n - 1\) edges is a tree.

Conversely, let \(T\) be a tree with order \(n\) and degree sequence \((d_1, d_2, \ldots, d_n)\). Since \(T\) is a tree, it is connected and so \(d_i > 0\) for all \(i = 1, \ldots, n\). We will show that the degree sum must be \(2(n - 1)\) by induction.

Consider the tree on two vertices. Since a tree is connected, there is necessarily one edge incident to these vertices, so the degree sequence is \(\{1, 1\}\) and \(d_1 + d_2 = 2 = 2(2 - 1)\).

For a tree on \(n\) vertices, assume the degree sum is equal to \(2(n - 1)\). We want to show that for a tree on \(n + 1\) vertices, the degree sum is \(2n\).

Suppose we have a tree on \(n + 1\) vertices. If we remove one pendant vertex, we decrease the degree sum by 2 and the number of edges by 1. Since this graph now has \(n\) vertices, then by the induction hypothesis, the have a degree sum of \(d_1 + d_2 + \ldots, d_k - 1, \ldots, d_n = 2(n - 1)\) where vertex \(k\) is the one adjacent to \(d_{n+1}\) in the graph of order \(n + 1\). If we append this vertex to this tree on \(n\) vertices to the vertex of degree \(d_k - 1\), we necessarily need one edge to do so, otherwise we would be creating a cycle. Therefore we would be increasing the degree sum by one for each of the two vertices, making the sum \(d_1 + d_2 + \ldots, d_n + d_{n+1} = 2(n - 1) + 2 = 2n\).

58. A forest is a graph each of whose connected components is a tree. In particular, a tree is a forest. Prove that a graph is a forest if and only if it does not have any cycles.

Since a forest with more than one component is a collection of trees and is disjoint, no cycle can exist that contains vertices of different components since there is no path between vertices of different components. So it suffices to show this statement is true for a single tree, which is by definition a forest.

Let \(T\) be a forest with one component and let \(u\) and \(v\) be any vertices of this component. Since they are in the same component, there must be a \(u - v\) path in \(T\). Suppose that there are two distinct \(u - v\) paths in \(T\), say \(P_1\) and \(P_2\). Then, we could traverse \(P_1\) from \(u\) to \(v\) and then return to \(u\) from \(v\) via \(P_2\). The removal of any edge of this cycle would not disconnect the graph, making no edge on this path a bridge. This contradicts that all edges of a tree are a bridge and so there cannot be two distinct paths. Therefore, since \(T\) is a tree, there are no cycles.

Now suppose that a graph \(T\) has no cycles. Then there is only one path containing vertices \(u\) and \(v\). The removal of any edge on this path makes it so that there is no path in \(T\) that allows for a walk from \(u\) to \(v\), making all edges on the \(u - v\) path a bridge. A connected graph in which every edge is a bridge is necessarily a tree, so a graph with no cycles is a tree, and if this property is true for all connected components, then we have a forest.
59. Prove that the removal of an edge from a tree leaves a forest of two trees.

Suppose \( T \) is a tree on \( n \) vertices. \( T \) must have \( n-1 \) edges. Let \( e \) be an edge of \( T \). Since all edges of a tree are bridges, \( T' = T - e \) is disconnected.

Suppose that the removal of \( e \) creates a graph with at least 3 trees. Each of these trees has at least two pendent vertices. So, if we add an edge to \( T' \), we can only have two vertices incident to it, i.e. we could only connect two of the trees. This would leave the graph disconnected, which contradicts that \( T \) was initially a tree. Therefore, \( T' \) can only have two components. That is, \( T' \) is comprised of two trees.

60. Let \( G \) be a forest of \( k \) trees. What is the fewest number of edges that can be inserted in \( G \) in order to obtain a tree?

If we have \( k \) trees, the minimum number of edges needed to connect the trees is \( k - 1 \).

62. Prove that, if a tree has a vertex of degree \( p \), then it has at least \( p \) pendent vertices.

Let \( (d_1, d_2, \ldots, d_n) \) be the degree sequence of a tree of order \( n \). Since a tree is connected, we have \( d_1 \geq d_2 \geq \ldots \geq d_n \geq 1 \). Let \( k \) be the number of pendent vertices. Then \( d_i = 1 \) for \( i = n - k + 1, n - k + 2, \ldots, n - k + k \) and \( d_i \geq 2 \) for all \( i = 1, \ldots, n - k \). Then, we have

\[
2(n - 1) = d_1 + d_2 + \ldots + d_{n-k} + d_{n-k+1} + \ldots + d_n \\
= d_1 + (d_2 + \ldots + d_{n-k}) + k \\
\geq d_1 + 2(n - k - 1) + k \\
= d_1 + 2(n - 1) - k
\]

(Note: \( d_2 + \ldots d_{n-k} \geq 2(n - k + 1) \) since the minimum degree for those \( n - k + 1 \) vertices is 2)

From \( 2(n - 1) \geq d_1 + 2(n - 1 - k) \) we get \( k \geq d_1 \). Now, for any vertex \( v \) with \( deg(v) = p \), we have \( k \geq d_1 \geq p \), or that there are at least \( p \) pendent vertices.
63. Determine a spanning tree for each of the graphs in Figures 11.15 through 11.17.

Figure 11.15

Figure 11.16

Figure 11.17
65. Use the algorithm for a spanning tree in Section 11.5 to construct a spanning tree of the graph of the dodecahedron.