§ 8.6 Coloring Graphs
History of Graph Colorings

Theorem

*The Four Color Theorem*: Any planar graph can be colored in at most four colors.

But, it wasn’t always the Four Color Theorem ...
1852 Francis Guthrie
Four Color Conjecture

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- Augustus DeMorgan
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- 1879 Arthur Cayley credit to DeMorgan
- 1879 Alfred Kempe proof
- 1880 Peter Guthrie Tait proof
- 1890 Percy Heawood disproves Kempe, proves Five Color Theorem
Four Color Conjecture

- 1891 Julius Petersen disproves Tait

The Petersen graph is an example of a snark.
Four Color Conjecture

- 1891 Julius Petersen disproves Tait

The Petersen graph is an example of a **snark**.

**Definition**

A **snark** is a connected, bridgeless cubic graph with chromatic index 4. **Chromatic index** is analogous to chromatic number but deals with edge colorings.
Four Color Conjecture

- 1940’s Danilo Blanuša's new snarks
Four Color Theorem

- 1960’s and 1970’s Heinrich Heesch computer work
Four Color Theorem

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- 1976 Kenneth Appel and Wolfgang Haken proof requiring various computer applications
- 1980’s Ulrich Schmidt finds error
- 1989 Appel and Hakem publish book addressing errors
- 2005 Benjamin Watson and Georges Gonthier proof with only one necessary program
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Chromatic Number $\chi(G)$

Definition

A coloring of a graph $G$ is an assignment of colors to the vertices of $G$. 
Chromatic Number $\chi(G)$

**Definition**
A coloring of a graph $G$ is an assignment of colors to the vertices of $G$.

**Example**
Three colorings on $C_4$.

![Colorings of $C_4$](image)
Proper Colorings

Definition
A proper coloring exists when adjacent vertices are colored differently.

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Proper Colorings

**Definition**

A proper coloring exists when adjacent vertices are colored differently.

**Example**

- The smallest integer $k$ such that a proper coloring of $G$ exists is called the chromatic number and is denoted by $\chi(G)$. 
Chromatic Number

What is the chromatic number for ...

- $C_4$?
Chromatic Number

What is the chromatic number for ...

- $C_4$?
- $C_5$?
Chromatic Number

What is the chromatic number for ...

- $C_4$?
- $C_5$?
- $C_{2n}$?
What is the chromatic number for ...

- $C_4$?
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- $C_{2n+1}$?
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- $C_4$?
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- $K_n$?
Chromatic Number

What is the chromatic number for ...

- $C_4$?
- $C_5$?
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- $K_n$?
- Bipartite graph?
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- $C_4$?
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- $K_n$?
- Bipartite graph?
- Tree?
Chromatic Number

- Wheel \( W_{2n} \)?
Chromatic Number

- Wheel $W_{2n}$?

- Wheel $W_{2n+1}$?
Example

What is $\chi(G)$ for the given graph?
Example

Find $\chi(G)$ for the given graph.

$G$
Greedy Algorithm for Vertex Coloring

Greedy Algorithm

Greedy Algorithms work by making the decision that seems most promising at any moment; it never reconsiders this decision, whatever situation may arise later.
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The Greedy Algorithm

Given a graph $G(V, E)$ with vertex set $V = \{v_1, \ldots, v_n\}$ and adjacency list $A_{v_j}$, construct a function $c : V \to \mathbb{Z}^+$ such that if $e = \{v_i, v_j\}$ then $c(v_1) \neq c(v_j)$.

1. Set $c(v_j) = 0$ for all $1 \leq j \leq n$
2. Set $c(v_1) = 1$
3. For $2 \leq j \leq n$, choose a color $k > 0$ for each vertex $v_j$ that differs from its neighbors by

$$c(v_j) = \min\{k \in \mathbb{Z}^+ | k > 0, c(w) \neq k \forall w \in A_{v_j}\}$$
Find the chromatic number for the following graph using the Greedy Algorithm.
which corresponds to
Example

Find the chromatic number for the following graph using the Greedy Algorithm.
Solution
Theorem

Brook’s Theorem (1941) If G is connected then if G is not an odd cycle nor complete then $\chi(G) \leq \Delta(G)$. 
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**Definition**

A graph is *$k$-colorable* if a proper coloring exists on $k$ colors.
Theorems

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*Vizing’s Theorem* (1964): Every undirected graph $G$ can be colored using a number of colors that is at most one larger than $\Delta(G)$. 
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**Theorem**

Konig (1936) A graph is 2-colorable iff it has no circuits of odd length.
Example

Suppose the Dean wants to schedule committee meetings and wants to schedule them in as few time slots as possible. How many would be needed if a meeting cannot take place at the same time as another meeting if the committees have any common members.

Curriculum Committee: A,B,F
Athletics: D,E,F
Academic Affairs: C,D,G
Chairs Meeting: A,E,G
Student Senate: B,D,E
We have to decide which are the vertices and which are the edges.
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\[ \chi(G) = 4 \]
Chromatic Polynomials

Definition

$P_G(x)$ is the chromatic polynomial for the graph $G$. It counts the number of ways to color $G$ in at most $x$ colors.
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Example

Find $P_{K_3}(x)$ and find out how many colorings there are on 1, 3 and 5 colors.
Solution

\[ P_{K_3}(x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x \]
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- For 1 color, there are 0 proper colorings
- For 3 colors, there are 6 proper colorings
- For 5 colors, there are 60 proper colorings
Solution

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Properties of Chromatic Polynomials

1. Degree is $n$

2. The lead coefficient is 1.

3. The constant term is 0.

4. $x$ is always a factor.

5. Either $P_G(x) = x^n$ or the sum of the coefficients is 0.

6. $\chi(G) = k$ is the smallest integer such that $P_G(k) \neq 0$.

7. The absolute value of the coefficient of the $x^{n-1}$ term is the size of the graph.
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Example

Find $P_{K_4}(x)$ and $P_{K_5}(x)$. 
Solution for $K_4$

$K_4$
Solution for $K_4$

$$P_{K_4}(x) = x(x - 1)(x - 2)(x - 3)$$
Solution for $K_5$
Solution for $K_5$

$$P_{K_5}(x) = x(x - 1)(x - 2)(x - 3)(x - 4)$$
Solution for $K_5$

$P_{K_5}(x) = x(x - 1)(x - 2)(x - 3)(x - 4)$

Pattern?
Solution for $K_5$

$$P_{K_5}(x) = x(x - 1)(x - 2)(x - 3)(x - 4)$$

Pattern? $P_{K_n}(x) = x(x - 1)(x - 2) \cdots (x - (n - 1))$
Example

Find $P_G(x)$ for the following graph.

$P_G(x) = x(x-1)^5$

Chromatic Polynomial for a Tree

$P_T(n) = x(x-1)^n - 1$
Example

Find $P_G(x)$ for the following graph.

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Example

Find $P_G(x)$ for the following graph.

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Chromatic Polynomial for a Tree

$P_{T_n}(x) = x(x - 1)^{n-1}$
Example

Find $P_{C_4}(x)$

\[ a \bullet - \bullet b \]

\[ d \bullet - \bullet c \]
There are two different cases to consider:

Case 1:

- $x$ choices for $a$
- $b$ and $d$ get the same color since they are nonadjacent, so $x - 1$ choices
- $x - 1$ choices for $c$, since it is adjacent to vertices with the same color

But, remember ... we are looking for number of proper colorings, not just chromatic number - we already know that.
Solution

There are two different cases to consider:

Case 1:

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- $x - 1$ choices for $c$, since it is adjacent to vertices with the same color

But, remember ... we are looking for number of proper colorings, not just chromatic number - we already know that.

Case 2:

- $x$ choices for $a$
- $x - 1$ choices for $b$
- We force $d$ to be a different color than $b$, so there are $x - 2$ choices
- $x - 2$ choices for $c$

In math, ‘or’ means $+$, so we have

$$P_{C_4}(x) = x(x - 1)(x - 2)^2 + x(x - 1)^2 = x^4 - 4x^3 + 6x^2 - 3x$$
Example

Find $P_G(x)$ for the following graph.
Solution

\[ P_G(x) = x(x - 1)(x - 2)^2(x - 3) \]
Two Piece Theorem

**Theorem**

Two Piece Theorem Let the vertex set of $G$ be partitioned into two disjoint sets $w_1$ and $w_2$ and let $G_1$ and $G_2$ be the subgraphs generated by $w_1$ and $w_2$, respectively. Suppose that in $G_1$, no edge joins a vertex of $w_1$ to a vertex of $w_2$. Then

$$P_G(x) = P_{G_1}(x)P_{G_2}(x)$$
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**Example**

Find $P_G(x)$ for the given graph.
Solution

Example

Find $P_G(x)$ for the given graph.

\[ P_{G_1}(x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x \]
\[ P_{G_2}(x) = x(x - 1)(x - 2)^2 = x^4 - 5x^3 + 8x^2 - 4x \]
\[ P_G(x) = (x^3 - 3x^2 + 2x)(x^4 - 5x^3 + 8x^2 - 4x) \]
The Fundamental Reduction Theorem

The Fundamental Reduction Theorem

\[ P_G(x) = P_{G'_\alpha}(x) - P_{G''_\alpha}(x) \]

where \( G'_\alpha = G - \alpha \) is a deletion and \( G''_\alpha \) is a contraction.
Example

Find $P_{C_4}(x)$ using the Fundamental Reduction Theorem.
Example

Find $P_{C_4}(x)$ using the Fundamental Reduction Theorem.

\[ P_{G'_\alpha}(x) = x(x - 1)^3 = x^4 - 3x^3 + 3x^2 - x \]
\[ P_{G''_\alpha}(x) = x(x - 1)(x - 2) = x^3 - 3x^2 + 2x \]
\[ P_G(x) = x^4 - 3x^3 + 3x^2 - x - (x^3 - 3x^2 + 2x) \]
\[ = x^4 - 4x^3 + 6x^2 - 3x \]
Use the Fundamental Reduction Theorem to find the chromatic polynomial for the given graph.

\[ G \]
Multiple choices of $\alpha$ would work here.
Solution

Multiple choices of $\alpha$ would work here.

So now we have ...

$$P_{G'_\alpha}(x) = x(x - 1)^3 = x^4 - 3x^3 + 3x^2 - x$$
Solution (cont.)

\[ P_{G''_{\alpha}}(x) = x(x - 1)^2 = x^3 - 2x^2 + x \]
Solution (cont.)

\[ P_{G''}(x) = x(x - 1)^2 = x^3 - 2x^2 + x \]

Putting this together, we have

\[ P_G(x) = x^4 - 3x^3 + 3x^2 - x - (x^3 - 2x^2 + x) \]
\[ = x^4 - 4x^3 + 5x^2 - 2x \]
Theorem

Suppose we have a set $A$, $|A| = N$. If the elements of $A$ have the properties $a_1, a_2, \ldots, a_p$ then the number of objects having none of these properties is given by

$$N(a'_1 \cdots a'_p) = N - \sum_i N(a_i) + \sum_{i \neq j} N(a_i a_j) \pm \ldots + (-1)^p N(a_1 a_2 \cdots a_p)$$
Example

How many integers between 1 and 1000 are not divisible by 3, 5 or 7?
Solution

For our properties, we have

- $a_1$ is the property the integer is divisible by 3
- $a_2$ is the property the integer is divisible by 5
- $a_3$ is the property the integer is divisible by 7
Solution

For our properties, we have

- $a_1$ is the property the integer is divisible by 3
- $a_2$ is the property the integer is divisible by 5
- $a_3$ is the property the integer is divisible by 7

\[
N(a_1) = \left\lfloor \frac{1000}{3} \right\rfloor = 333
\]
\[
N(a_2) = \left\lfloor \frac{1000}{5} \right\rfloor = 200
\]
\[
N(a_3) = \left\lfloor \frac{1000}{7} \right\rfloor = 142
\]

So, by the Principle of Inclusion-Exclusion, we have

\[
N(a_1a_2a_3) = 1000 - (333 + 200 + 142) + (66 + 47 + 28) - 9 = 457
\]
Solution

For our properties, we have

\( a_1 \) is the property the integer is divisible by 3
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\[
N(a_1 a_2) = \left\lfloor \frac{1000}{15} \right\rfloor = 66
\]
\[
N(a_1 a_3) = \left\lfloor \frac{1000}{21} \right\rfloor = 47
\]
\[
N(a_2 a_3) = \left\lfloor \frac{1000}{35} \right\rfloor = 28
\]

So, by the Principle of Inclusion-Exclusion, we have

\[
N(a_1' a_2' a_3') = 1000 - (333 + 200 + 142) + (66 + 47 + 28) - 9 = 457
\]
Solution

For our properties, we have

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N(a_1a_2a_3) = \left\lfloor \frac{1000}{105} \right\rfloor = 9
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Solution

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N(a_1a_2a_3) = \left\lfloor \frac{1000}{105} \right\rfloor = 9
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So, by the Principle of Inclusion-Exclusion, we have

\[
N(a'_1a'_2a'_3) = 1000 - (333 + 200 + 142) + (66 + 47 + 28) - 9 = 457
\]
Example

Find $P_{C_4}(x)$ using the Principle of Inclusion-Exclusion.

\[\begin{align*}
\text{Example} \\
\text{Find } P_{C_4}(x) \text{ using the Principle of Inclusion-Exclusion.}
\end{align*}\]
Solution

Let $a_i$ be the property that edge $e_i$ has both incident vertices colored with the same color. So, what we will want is none of the $a_i$ to be true. That is, we are looking for $N(a'_1a'_2a'_3a'_4)$.

\[ N = x^4 \]
Let $a_i$ be the property that edge $e_i$ has both incident vertices colored with the same color. So, what we will want is none of the $a_i$ to be true. That is, we are looking for $N(a'_1a'_2a'_3a'_4)$.

$$N = x^4$$

$$N(a_1) = x^3$$
Solution

Let $a_i$ be the property that edge $e_i$ has both incident vertices colored with the same color. So, what we will want is none of the $a_i$ to be true. That is, we are looking for $N(a'_1 a'_2 a'_3 a'_4)$.

$$N = x^4$$

$$N(a_1) = x^3 \Rightarrow \sum N(a_i) = 4x^3$$
Let $a_i$ be the property that edge $e_i$ has both incident vertices colored with the same color. So, what we will want is none of the $a_i$ to be true. That is, we are looking for $N(a'_1 a'_2 a'_3 a'_4)$.

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N(a_1 a_2) = x^2
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Let $a_i$ be the property that edge $e_i$ has both incident vertices colored with the same color. So, what we will want is none of the $a_i$ to be true. That is, we are looking for $N(a'_1a'_2a'_3a'_4)$.

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$$N(a_1) = x^3 \Rightarrow \sum N(a_i) = 4x^3$$

$$N(a_1a_2) = x^2 \Rightarrow \sum N(a_i a_j) = 6x^2$$
Solution

Let $a_i$ be the property that edge $e_i$ has both incident vertices colored with the same color. So, what we will want is none of the $a_i$ to be true. That is, we are looking for $N(a_1' a_2' a_3' a_4')$.

$$N = x^4$$

$$N(a_1) = x^3 \Rightarrow \sum N(a_i) = 4x^3$$

$$N(a_1 a_2) = x^2 \Rightarrow \sum N(a_i a_j) = 6x^2$$

$$N(a_1 a_2 a_3) = x$$
Solution

Let $a_i$ be the property that edge $e_i$ has both incident vertices colored with the same color. So, what we will want is none of the $a_i$ to be true. That is, we are looking for $N(a'_1 a'_2 a'_3 a'_4)$.

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$$N(a_1) = x^3 \Rightarrow \sum N(a_i) = 4x^3$$

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$$N(a_1 a_2 a_3) = x \Rightarrow \sum N(a_i a_j a_k) = 4x$$
Let $a_i$ be the property that edge $e_i$ has both incident vertices colored with the same color. So, what we will want is none of the $a_i$ to be true. That is, we are looking for $N(a'_1 a'_2 a'_3 a'_4)$.

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N(a_1) = x^3 \Rightarrow \sum N(a_i) = 4x^3
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N(a_1 a_2 a_3 a_4) = x
\]
Let \( a_i \) be the property that edge \( e_i \) has both incident vertices colored with the same color. So, what we will want is none of the \( a_i \) to be true. That is, we are looking for \( N(a'_1a'_2a'_3a'_4) \).

\[
N = x^4 \\
N(a_1) = x^3 \Rightarrow \sum N(a_i) = 4x^3 \\
N(a_1a_2) = x^2 \Rightarrow \sum N(a_ia_j) = 6x^2 \\
N(a_1a_2a_3) = x \Rightarrow \sum N(a_ia_ja_k) = 4x \\
N(a_1a_2a_3a_4) = x
\]

Therefore, we have

\[
P_{C_4}(x) = N(a'_1a'_2a'_3a'_4) = x^4 - 4x^3 + 6x^2 - 4x + x = x^4 - 4x^3 + 6x^2 - 3x
\]
Example

Find $P_G(x)$ for the following graph by using the Principle of Inclusion-Exclusion.
Solution

\[ N = \]

\[ N(a_1) = \]

\[ \sum_i N(a_i) = 5 \]

\[ N(a_1 a_2) = \]

\[ \sum_{i \neq j} N(a_i a_j) = 10 \]

\[ N(a_1 a_2 a_3) = \]

\[ \sum_{i \neq j \neq k} N(a_i a_j a_k) = 2 + 8 \]
Solution

\[ N = x^4 \]

\[ N(a_1) = \]
Solution

\[ N = x^4 \]

\[ N(a_1) = x^3 \Rightarrow \sum_{i} N(a_i) = \]
Solution

\[ N = x^4 \]

\[ N(a_1) = x^3 \Rightarrow \sum_i N(a_i) = 5x^3 \]

\[ N(a_1 a_2) = \]
Solution

\[ N = x^4 \]

\[ N(a_1) = x^3 \Rightarrow \sum_i N(a_i) = 5x^3 \]

\[ N(a_1a_2) = x^2 \Rightarrow \sum_{i \neq j} N(a_ia_j) = \]
Solution

\[ N = x^4 \]

\[ N(a_1) = x^3 \Rightarrow \sum_{i} N(a_i) = 5x^3 \]

\[ N(a_1a_2) = x^2 \Rightarrow \sum_{i \neq j} N(a_ia_j) = 10x^2 \]

\[ N(a_1a_2a_3) = \]
Solution

\[ N = x^4 \]

\[ N(a_1) = x^3 \Rightarrow \sum_i N(a_i) = 5x^3 \]

\[ N(a_1a_2) = x^2 \Rightarrow \sum_{i \neq j} N(a_ia_j) = 10x^2 \]

\[ N(a_1a_2a_3) = x^2 \Rightarrow \sum_{i \neq j \neq k} N(a_ia_ja_k) = \]
Solution

\[ N = x^4 \]

\[ N(a_1) = x^3 \Rightarrow \sum_i N(a_i) = 5x^3 \]

\[ N(a_1a_2) = x^2 \Rightarrow \sum_{i \neq j} N(a_ia_j) = 10x^2 \]

\[ N(a_1a_2a_3) = x^2 \Rightarrow \sum_{i \neq j \neq k} N(a_ia_ja_k) = 2x^2 + 8x \]
Solution

\[ N(a_1a_2a_3a_4) = \]

\[ P_G(x) = x^4 - 5x^3 + 8x^2 - 4x \]
Solution

\[ N(a_1a_2a_3a_4) = x \Rightarrow \sum_{i \neq j \neq k \neq l} N(a_ia_ja_ka_l) = \]

So,

\[ P_G(x) = x^4 - 5x^3 + 8x^2 - 4x \]
Solution

\[ N(a_1 a_2 a_3 a_4) = x \implies \sum_{i \neq j \neq k \neq l} N(a_i a_j a_k a_l) = 5x \]

\[ N(a_1 a_2 a_3 a_4 a_5) = \]
Solution

\[ N(a_1a_2a_3a_4) = x \Rightarrow \sum_{i \neq j \neq k \neq l} N(a_i a_j a_k a_l) = 5x \]

\[ N(a_1a_2a_3a_4a_5) = x \]
Solution

\[ N(a_1 a_2 a_3 a_4) = x \Rightarrow \sum_{i \neq j \neq k \neq l} N(a_i a_j a_k a_l) = 5x \]

\[ N(a_1 a_2 a_3 a_4 a_5) = x \]

Therefore,

\[ N(a'_1 a'_2 a'_3 a'_4 a'_5) = x^4 - 5x^3 + 10x^2 - (2x^2 + 8x) + 5x - x \]

So,

\[ P_G(x) = x^4 - 5x^3 + 8x^2 - 4x \]