§ 8.2 Eulerian Paths and Circuits
Königsberg, 1736
Graph Representation
Do You Remember ...

Definition

A $u − v$ trail is a $u − v$ walk where no edge is repeated.
Definition

A $u - v$ trail is a $u - v$ walk where no edge is repeated.

Definition

A circuit is a nontrivial closed trail.
New Definitions

Definition
An Eulerian trail is an open trail of $G$ containing all edges and vertices.
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An Eulerian circuit is a closed trail containing all edges and vertices.
New Definitions

**Definition**

An **Eulerian trail** is an open trail of $G$ containing all edges and vertices.

**Definition**

An **Eulerian circuit** is a closed trail containing all edges and vertices.

**Definition**

A graph containing an Eulerian circuit is called **Eulerian**.
Examples

Which contain Eulerian trails? circuits?

Do you see any patterns as to when one does/does not exist?
More Examples
Conclusions

1. If all vertices are even, an Eulerian circuit exists

2. If there are exactly 2 odd vertices, an Eulerian trail exists that begins at one odd vertex and ends at the other one

3. If there are more than two odd vertices, no Eulerian trail or circuit exists
Conclusions

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3. If there are more than two odd vertices, no Eulerian trail or circuit exists
Theorem

A graph $G$ contains an Eulerian circuit if and and only if the degree of each vertex is even.
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Proof of Necessity

Suppose $G$ contains an Eulerian circuit $C$. Then, for any choice of vertex $v$, $C$ contains all the edges that are incident to $v$. Furthermore, as we traverse along $C$, we must enter and leave $v$ the same number of times, and it follows that $\text{deg}(v)$ must be even.
Example

\[ K_5 \text{ is 4-regular and so has all even vertices.} \]
Euler presented proof of necessity in 1736
Notes

- Euler presented proof of necessity in 1736
- His paper did not include the proof of the converse
Euler presented proof of necessity in 1736
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Proof of sufficiency was presented in 1873 by a German mathematician named Carl Hierholzer
Proof of Sufficiency

We prove by induction on the number of edges. For graphs with all vertices of even degree, the smallest possible number of edges is 3 in the case of simple graphs, and 2 in the case of multigraphs. In both cases, the graph trivially contains an Eulerian circuit.
Proof of Sufficiency

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Let $H$ be a connected graph with $k$ edges. If every vertex of $H$ has even degree, $H$ contains an Eulerian circuit.
Proof of Sufficiency

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\textit{Let H be a connected graph with k edges. If every vertex of H has even degree, H contains an Eulerian circuit.}

Now, let G be a graph with \( k + 1 \) edges, and every vertex has an even degree. Since there is no odd degree vertex, G cannot be a tree (acyclic, connected graph). Thus, G must contain a cycle C.
Now, remove the edges of $C$ from $G$, and consider the remaining graph $G'$. Since removing $C$ from $G$ may disconnect the graph, $G'$ is a collection of connected components, namely $G_1, G_2, \ldots$. 
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Furthermore, when the edges in $C$ are removed from $G$, each vertex loses even number of adjacent edges. Thus, the parity of each vertex is unchanged in $G'$. It follows that, for each connected component of $G'$, every vertex has an even degree.
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Therefore, by the induction hypothesis, each of $G_1, G_2, \ldots$ has its own Eulerian circuit, namely $C_1, C_2, \ldots$. 
We can now build an Eulerian circuit for $G$. Pick an arbitrary vertex $a$ from $C$. Traverse along $C$ until we reach a vertex $v_i$ that belongs to one of the connected components $G_i$. 
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Then, traverse along its Eulerian circuit $C_i$ until we traverse all the edges of $C_i$. We are now back at $v_i$, and so we can continue on along $C$. In the end, we shall return back to the first starting vertex $a$, after visiting every edge exactly once.
Illustration of Theorem
Illustration of Theorem
Illustration of Theorem (cont..)
Illustration of Theorem (cont..)
Illustration of Theorem (cont..)
Eulerian trail

**Theorem**

A graph contains an Eulerian path if and only if there are 0 or 2 odd degree vertices.
Eulerian trail

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Suppose a graph $G$ contains an Eulerian path $P$. Then, for every vertex $v$, $P$ must enter and leave $v$ the same number of times, except when it is either the starting vertex or the final vertex of $P$. When the starting and final vertices are distinct, there are precisely 2 odd degree vertices. When these two vertices coincide, there is no odd degree vertex.

Conversely, suppose $G$ contains 2 odd degree vertices $u$ and $v$. (The case where $G$ has no odd degree vertex is shown in the previous theorem.) Then, temporarily add a dummy edge $\{u, v\}$ to $G$. Now the modified graph contains no odd degree vertex. By the last theorem, this graph contains an Eulerian circuit $C$ that also contains $\{u, v\}$. Remove $\{u, v\}$ from $C$, and now we have an Eulerian path where $u$ and $v$ serve as initial and final vertices.
Eulerian trail

**Theorem**

A graph contains an Eulerian path if and only if there are 0 or 2 odd degree vertices.

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<th>Kwan, 1962</th>
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Edges must be added to create a multigraph because the simple graph contains no Eulerian circuit.