Chapter 6 Eigenvalues and Eigenvectors

6.1 Eigenvalues and Eigenvectors
What are eigenvalues and eigenvectors?

Definition

Given an \( n \times n \) matrix \( A \), a nonzero vector \( \mathbf{u} \) in \( \mathbb{R}^n \) is called an eigenvector of \( A \) if there exists a scalar \( \lambda \) such that \( A\mathbf{u} = \lambda \mathbf{u} \). The scalar \( \lambda \) is called an eigenvalue of \( A \).
What are eigenvalues and eigenvalues?

**Definition**

Given an $n \times n$ matrix $A$, a nonzero vector $u$ in $\mathbb{R}^n$ is called an **eigenvector** of $A$ if there exists a scalar $\lambda$ such that $Au = \lambda u$. The scalar $\lambda$ is called an **eigenvalue** of $A$.

Note: $\lambda$ is allowed to be 0.
Why are eigenvalues and eigenvectors important?

Tacoma Narrows Bridge, 1940
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Does anyone know what the natural frequency of an object like a bridge is?
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Definition

The natural frequency is the frequency at which a system naturally vibrates once it has been set into motion. That is, it is motion a structure takes on in response to wind or being walked on.
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The reason this bridge collapsed is because the natural frequency of the bridge was too close to the natural frequency of the wind. When frequencies match, the compound provided too strong a force for the bridge.
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Mathematically, the natural frequency can be characterized by the eigenvalue of smallest magnitude.
Why are eigenvalues and eigenvectors important?

Linear transformations: Eigenvectors make understanding linear transformations easy. They are the “axes” (directions) along which a linear transformation acts simply by stretching/compressing and/or reflecting.
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Eigenvalues give you the factors by which this compression occurs.

The more directions you have along which you understand the behavior of a linear transformation, the easier it is to understand the linear transformation; so you want to have as many linearly independent eigenvectors as possible associated to a single linear transformation.
Why are eigenvalues and eigenvectors important?

Eigenvalues tell us a whole lot about a matrix:

1. If \( A \) is singular then \( \lambda = 0 \)
2. If \( A \) is symmetric, then all eigenvalues are real
3. If \( A \) has rank \( n \) then the eigenvectors form a basis for \( \mathbb{R}^n \)
4. The eigenvectors of \( AA^T \) form a basis for \( \text{col}(A) \)
5. The eigenvectors of \( A^TA \) form a basis for \( \text{row}(A) \)
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How these work

Example

Verify that \( \mathbf{u} \) is an eigenvector for the matrix \( A \) and determine the associated eigenvalue.

\[
A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]
How these work

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\[
\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}
\]

Therefore, we have shown \( \mathbf{u} \) is an eigenvector of \( A \) and \( \lambda = 4 \) is the associated eigenvalue.
How these work

Example

Verify that \( u \) is an eigenvector for the matrix \( A \) and determine the associated eigenvalue.

\[
A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
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Visualization
Visualization
Another Example

Example
Verify that \( \mathbf{u} \) is an eigenvector for the matrix \( A \) and determine the associated eigenvalue.

\[
A = \begin{bmatrix}
10 & 12 & -6 \\
8 & 11 & -4 \\
34 & 44 & -19
\end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix}
0 \\
1 \\
2
\end{bmatrix}
\]

Therefore, we have shown \( \mathbf{u} \) is an eigenvector of \( A \) and \( \lambda = 3 \) is the associated eigenvalue.
Another Example

Example

Verify that \( \mathbf{u} \) is an eigenvector for the matrix \( A \) and determine the associated eigenvalue.

\[
A = \begin{bmatrix} 10 & 12 & -6 \\ 8 & 11 & -4 \\ 34 & 44 & -19 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}
\]

\[
\begin{bmatrix} 10 & 12 & -6 \\ 8 & 11 & -4 \\ 34 & 44 & -19 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}
\]

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Therefore, we have shown that \( \mathbf{u} \) is an eigenvector of \( A \) and \( \lambda = 3 \) is the associated eigenvalue.
Another Example

Example

Verify that $\mathbf{u}$ is an eigenvector for the matrix $A$ and determine the associated eigenvalue.

$$A = \begin{bmatrix} 10 & 12 & -6 \\ 8 & 11 & -4 \\ 34 & 44 & -19 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 12 & -6 \\ 8 & 11 & -4 \\ 34 & 44 & -19 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Therefore, we have shown $\mathbf{u}$ is an eigenvector of $A$ and $\lambda = 3$ is the associated eigenvalue.
How we find eigenvalues

Example

Find the characteristic polynomial and eigenvalues for $A$.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$
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The characteristic polynomial is found by finding $\det(A - \lambda I_n)$, so in this case we need $\det(A - \lambda I_2)$,
How we find eigenvalues

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The characteristic polynomial is found by finding \( \det(A - \lambda I_n) \), so in this case we need \( \det(A - \lambda I_2) \),

\[
\det(A - \lambda I_2) = \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix}
\]
How we find eigenvalues

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$$\det(A - \lambda I_2) = \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^2 - 1$$
How we find eigenvalues

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Find the characteristic polynomial and eigenvalues for $A$.

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The characteristic polynomial is found by finding $\text{det}(A - \lambda I_n)$, so in this case we need $\text{det}(A - \lambda I_2)$,

$$\text{det}(A - \lambda I_2) = \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix}$$

$$= (3 - \lambda)^2 - 1$$

$$= 9 - 6\lambda + \lambda^2 - 1$$
How we find eigenvalues

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Find the characteristic polynomial and eigenvalues for $A$.

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$$= 9 - 6\lambda + \lambda^2 - 1$$

$$= \lambda^2 - 6\lambda + 8$$
How we find eigenvalues

So, our characteristic polynomial is $\lambda^2 - 6\lambda + 8$. 
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The eigenvalues are the values of $\lambda$ such that $\det(A - \lambda I_n) = 0$. 

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So, here we have

$$\lambda^2 - 6\lambda + 8 = 0$$
How we find eigenvalues

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The eigenvalues are the values of \( \lambda \) such that \( \det(A - \lambda I_n) = 0 \).

So, here we have

\[
\lambda^2 - 6\lambda + 8 = 0 \\
(\lambda - 2)(\lambda - 4) = 0
\]
How we find eigenvalues

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The eigenvalues are the values of $\lambda$ such that $\det(A - \lambda I_n) = 0$.

So, here we have

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 2)(\lambda - 4) = 0$$

$$\lambda = 2, 4$$
How we find eigenvalues

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The eigenvalues are the values of $\lambda$ such that $\det(A - \lambda I_n) = 0$.

So, here we have

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 2)(\lambda - 4) = 0$$

$\lambda = 2, 4$

So, the eigenvalues associated with $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ are $\lambda = 2$ and $\lambda = 4$. 
Another Example

Example

Find the characteristic polynomial and eigenvalues for $A$.

$$A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$$
Another Example

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Find the characteristic polynomial and eigenvalues for $A$.

$$A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} -7 - \lambda & 10 \\ -5 & 8 - \lambda \end{vmatrix}$$
Another Example

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Find the characteristic polynomial and eigenvalues for $A$.

$$A = \begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}$$

$$\det(A - \lambda I_2) = \begin{vmatrix} -7 - \lambda & 10 \\ -5 & 8 - \lambda \end{vmatrix} = (-7 - \lambda)(8 - \lambda) + 50$$
Another Example

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$$\det(A - \lambda I_2) = \begin{vmatrix} -7 - \lambda & 10 \\ -5 & 8 - \lambda \end{vmatrix}$$

$$= (-7 - \lambda)(8 - \lambda) + 50$$

$$= -56 - \lambda + \lambda^2 + 50$$
Another Example

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Find the characteristic polynomial and eigenvalues for $A$.

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$$= (-7 - \lambda)(8 - \lambda) + 50$$

$$= -56 - \lambda + \lambda^2 + 50$$

$$= \lambda^2 - \lambda - 6$$
Another Example

Now,

$$\lambda^2 - \lambda - 6 = 0$$
Another Example

Now,

\[\lambda^2 - \lambda - 6 = 0\]
\[(\lambda - 3)(\lambda + 2) = 0\]
Another Example

Now,

\[ \lambda^2 - \lambda - 6 = 0 \]
\[ (\lambda - 3)(\lambda + 2) = 0 \]
\[ \lambda = 3, -2 \]
Another Example

Now,

\[ \lambda^2 - \lambda - 6 = 0 \]
\[ (\lambda - 3)(\lambda + 2) = 0 \]
\[ \lambda = 3, -2 \]

Therefore, the eigenvalues associated with \[\begin{bmatrix} -7 & 10 \\ -5 & 8 \end{bmatrix}\] are \(\lambda = 3\) and \(\lambda = -2\).
Example

Find the characteristic polynomial and eigenvalues for $A$.

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$
One More

\[
\det(A - \lambda I_3) = \begin{vmatrix}
1 - \lambda & 0 & -2 \\
0 & 1 - \lambda & 2 \\
2 & 2 & 1 - \lambda \\
\end{vmatrix}
\]
One More

\[ \text{det}(A - \lambda I_3) = \begin{vmatrix} 1 - \lambda & 0 & -2 \\ 0 & 1 - \lambda & 2 \\ 2 & 2 & 1 - \lambda \end{vmatrix} \]

\[ = (1 - \lambda) \begin{vmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 - \lambda \\ 2 & 2 \end{vmatrix} \]

\[ = (1 - \lambda)(\lambda^2 - 2\lambda + 3) + 4(1 - \lambda) \]

\[ = -\lambda^3 + 3\lambda^2 - 3\lambda + 1 \]
One More

\[
\text{det}(A - \lambda I_3) = \begin{vmatrix} 1 - \lambda & 0 & -2 \\ 0 & 1 - \lambda & 2 \\ 2 & 2 & 1 - \lambda \end{vmatrix}
\]

\[
= (1 - \lambda) \begin{vmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 - \lambda \\ 2 & 2 \end{vmatrix}
\]

\[
= (1 - \lambda)((1 - \lambda)^2 - 4) - 2(0 - 2(1 - \lambda))
\]
One More

\[
\det(A - \lambda I_3) = \begin{vmatrix}
1 - \lambda & 0 & -2 \\
0 & 1 - \lambda & 2 \\
2 & 2 & 1 - \lambda
\end{vmatrix}
\]

\[
= (1 - \lambda)\begin{vmatrix}
1 - \lambda & 2 \\
2 & 1 - \lambda
\end{vmatrix} - 2\begin{vmatrix}
0 & 1 - \lambda \\
2 & 2
\end{vmatrix}
\]

\[
= (1 - \lambda)((1 - \lambda)^2 - 4) - 2(0 - 2(1 - \lambda))
\]

\[
= (1 - \lambda)(\lambda^2 - 2\lambda + 3) + 4(1 - \lambda)
\]
One More

$$\det(A - \lambda I_3) = \begin{vmatrix} 1 - \lambda & 0 & -2 \\ 0 & 1 - \lambda & 2 \\ 2 & 2 & 1 - \lambda \end{vmatrix}$$

$$= (1 - \lambda) \begin{vmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 1 - \lambda \\ 2 & 2 \end{vmatrix}$$

$$= (1 - \lambda)((1 - \lambda)^2 - 4) - 2(0 - 2(1 - \lambda))$$

$$= (1 - \lambda)(\lambda^2 - 2\lambda + 3) + 4(1 - \lambda)$$

$$= -\lambda^3 + 3\lambda^2 - 3\lambda + 1$$
One More

Now,

\[-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0\]
One More

Now,

\[-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0\]

\[(1 - \lambda)^3 = 0\]
Now,

\[-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0\]

\[(1 - \lambda)^3 = 0\]

\[\lambda = 1\]
One More

Now,

\[-\lambda^3 + 3\lambda^2 - 3\lambda + 1 = 0\]

\[(1 - \lambda)^3 = 0\]

\[\lambda = 1\]

So, the only eigenvalue associated with

\[
\begin{bmatrix}
1 & -0 & -2 \\
0 & 1 & 2 \\
2 & 2 & 1 \\
\end{bmatrix}
\]

is \(\lambda = 1\).
Finding the eigenvectors

Definition
If $\lambda$ is an eigenvalue for an $n \times n$ matrix $A$, the eigenspace of $\lambda$ is the solution set to the homogeneous linear system $(A - \lambda I_n)x = 0$, that is to say, the subspace $\ker(A - \lambda I_n)$. 
Finding the eigenvectors

Definition
If \( \lambda \) is an eigenvalue for an \( n \times n \) matrix \( A \), the eigenspace of \( \lambda \) is the solution set to the homogeneous linear system \( (A - \lambda I_n)x = 0 \), that is to say, the subspace \( \ker(A - \lambda I_n) \).

So, to find the vectors we use the matrix \( A - \lambda I_n \) and row reduce to find the form of the infinite number of solutions. The vector can be represented as a basis for the eigenspace associated with the given eigenvalue.
Finding the bases for the eigenspaces

Example

Find a basis for the eigenspace of $A$ associated with each eigenvalue.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$
Finding the bases for the eigenspaces

Example

Find a basis for the eigenspace of $A$ associated with each eigenvalue.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

From earlier, we found that the eigenvalues were $\lambda = 2$ and $\lambda = 4$. 
Finding the bases for the eigenspaces

$\lambda = 2$: Remember, we are solving the homogeneous system...

$$A - 2I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}$$
Finding the bases for the eigenspaces

$\lambda = 2$: Remember, we are solving the homogeneous system

$$A - 2I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$
Finding the bases for the eigenspaces

\( \lambda = 2 \): Remember, we are solving the homogeneous system...

\[
A - 2I_2 = \begin{bmatrix}
3 - \lambda & 1 \\
1 & 3 - \lambda
\end{bmatrix}
= \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}
\sim \begin{bmatrix}
1 & 1 \\
0 & 0
\end{bmatrix}
\]
Finding the bases for the eigenspaces

\( \lambda = 2 \): Remember, we are solving the homogeneous system...

\[
A - 2I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]

Solving this system, we get \( \mathbf{x} = s \begin{bmatrix} 1 \\ -1 \end{bmatrix} \).
Finding the bases for the eigenspaces

$\lambda = 2$: Remember, we are solving the homogeneous system...

\[
A - 2I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}
\]

Solving this system, we get $x = s \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

So, a basis for the eigenspace associated with $\lambda = 2$ is $\text{sp} \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$. 
Find the bases for the eigenspaces

$\lambda = 4$:

$$A - 4I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}$$

Solving this system, we get

$$x = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So, a basis for the eigenspace associated with $\lambda = 4$ is

$$\text{sp}\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$$
Find the bases for the eigenspaces

$$\lambda = 4$$:

$$A - 4I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

Solving this system, we get

$$x = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

So, a basis for the eigenspace associated with $$\lambda = 4$$ is

$$\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$$.
Find the bases for the eigenspaces

$\lambda = 4$:

\[
A - 4I_2 = \begin{bmatrix}
3 - \lambda & 1 \\
1 & 3 - \lambda \\
\end{bmatrix}
= \begin{bmatrix}
-1 & 1 \\
1 & -1 \\
\end{bmatrix}
\sim \begin{bmatrix}
1 & -1 \\
0 & 0 \\
\end{bmatrix}
\]

Solving this system, we get $x = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

So, a basis for the eigenspace associated with $\lambda = 4$ is $\text{span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.
Find the bases for the eigenspaces

\[\lambda = 4:\]

\[A - 4I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\]

Solving this system, we get \(x = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}\)
Find the bases for the eigenspaces

λ = 4:

\[
A - 4I_2 = \begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}
= \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}
\sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}
\]

Solving this system, we get \( x = s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \)

So, a basis for the eigenspace associated with \( \lambda = 4 \) is

\( \text{sp} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \).
The Big Theorem - Version 8

Theorem

Let \( A = \{a_1, \ldots, a_n\} \) be a set of \( n \) vectors in \( \mathbb{R}^n \), let
\[ A = \begin{bmatrix} a_1 & a_2 & \ldots & a_n \end{bmatrix} \]
and let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be given by \( T(x) = Ax \).

Then the following are equivalent:
The Big Theorem - Version 8

**Theorem**

Let \( \mathcal{A} = \{a_1, \ldots, a_n\} \) be a set of \( n \) vectors in \( \mathbb{R}^n \), let
\[
A = [a_1 \ a_2 \ \ldots \ \ a_n]
\]
and let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be given by \( T(x) = Ax \).
Then the following are equivalent:

(a) \( \mathcal{A} \) spans \( \mathbb{R}^n \)
(b) \( \mathcal{A} \) is linearly independent
(c) \( Ax = b \) has a unique solution \( \forall b \in \mathbb{R}^n \)
(d) \( T \) is onto
(e) \( T \) is 1-1
(f) \( A \) is invertible
(g) \( \ker(T) = \{0\} \)
(h) \( \mathcal{A} \) is a basis for \( \mathbb{R}^n \)
(i) \( \text{col}(A) = \mathbb{R}^n \)
(j) \( \text{row}(A) = \mathbb{R}^n \)
(k) \( \text{rank}(A) = n \)
(l) \( \det(A) \neq 0 \)
The Big Theorem - Version 8

Theorem

Let \( \mathcal{A} = \{a_1, \ldots, a_n\} \) be a set of \( n \) vectors in \( \mathbb{R}^n \), let \( A = [a_1 \ a_2 \ \ldots \ a_n] \) and let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be given by \( T(x) = Ax \).

Then the following are equivalent:

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(c) \( Ax = b \) has a unique solution \( \forall b \in \mathbb{R}^n \)
(d) \( T \) is onto
(e) \( T \) is 1-1
(f) \( \mathcal{A} \) is invertible
(g) \( \ker(T) = \{0\} \)
(h) \( \mathcal{A} \) is a basis for \( \mathbb{R}^n \)
(i) \( \text{col}(A) = \mathbb{R}^n \)
(j) \( \text{row}(A) = \mathbb{R}^n \)
(k) \( \text{rank}(A) = n \)
(l) \( \det(A) \neq 0 \)
(m) \( \lambda = 0 \) is not an eigenvalue of \( A \)